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# Introduction to Stokes Structures

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# Introduction

The classical theory of linear differential equation of one complex variable near a singular point distinguishes between a regular and an irregular singularity by checking the vanishing of the irregularity number, which characterizes a regular singularity (Fuchs criterion). The behaviour of the solutions of the equation (moderate growth near the singularity) also characterizes a regular singularity, and this leads to the local Riemann–Hilbert correspondence, characterizing a regular singularity by “monodromy data”.

On a Riemann surface  $X$ , the Riemann–Hilbert correspondence for meromorphic connections with regular singularities on a discrete set  $D$  (first case), or more generally for regular holonomic  $\mathcal{D}$ -modules with singularities at  $D$  (second case), induces an equivalence of the corresponding category with the category of “monodromy data”, which can be presented

- Either quiver-theoretically as the data of local monodromies and connection matrices (first case), together with the so-called canonical and variation morphisms (second case).
- Or sheaf-theoretically as the category of locally constant sheaves of finite dimensional  $\mathbb{C}$ -vector spaces on  $X^* = X \setminus D$  (first case) or perverse sheaves with singularities at  $D$  (second case).

While the first presentation is suited to describing moduli spaces, for instance, the second one is suited to sheaf-theoretic operations on such objects. Each of these objects can be defined over subfields  $k$  of  $\mathbb{C}$ , giving rise to a  $k$ -structure on the meromorphic connection with regular singularities, or regular holonomic  $\mathcal{D}$ -module.

When the irregularity number is nonzero, finer numerical invariants are introduced, encoded in the Newton polygon of the equation at the singular point. Moreover, such a Riemann–Hilbert correspondence with both aspects also exists. The first one is the most popular, with Stokes data, consisting of Stokes matrices, instead of local monodromy data. An extensive literature exists on this subject, for which classical references are [93, 96] and a more recent one is [94]. The second aspect, initiated by P. Deligne [17] (case of meromorphic connections) and [19]

(holonomic  $\mathcal{D}$ -modules), has also been developed by B. Malgrange [52, 55] and D. Babbitt and V.S. Varadarajan [2]. Moreover, the Poincaré duality has been expressed by integrals on “rapid decay cycles” by various authors [10, 33].

In higher dimensions, such a dichotomy (regular/irregular singularity) also exists for meromorphic bundles with flat connection (respectively holonomic  $\mathcal{D}$ -modules). The work of P. Deligne [15] has provided a notion of meromorphic connection with regular singularities, and a Riemann–Hilbert correspondence has been obtained by P. Deligne in such a case and by M. Kashiwara on the one hand and Z. Mebkhout on the other hand in the case of holonomic  $\mathcal{D}$ -modules with regular singularities. The target category for this correspondence is that of  $\mathbb{C}$ -perverse sheaves. Moreover, the Fuchs criterion has been generalized by Z. Mebkhout: the irregularity number is now replaced by the irregularity complex, which is also a perverse sheaf.

When the irregularity perverse sheaf is not zero, it can be refined, giving rise to Newton polygons on strata of a stratification adapted to the characteristic variety of the holonomic  $\mathcal{D}$ -module (see [44]).

These lectures will be mainly concerned with the second aspect of the Riemann–Hilbert correspondence for meromorphic connections or holonomic  $\mathcal{D}$ -modules, and the main keyword will be the Stokes phenomenon in higher dimension. Their purpose is to develop the original idea of P. Deligne and B. Malgrange and make it enter the frame of perverse sheaves, so that it can be extended to arbitrary dimensions. This has been motivated by recent beautiful results of T. Mochizuki [67, 70], who has rediscovered it and shown the powerfulness of this point of view in higher dimension.

This approach is intended to provide a global understanding of the Stokes phenomenon. While in dimension one the polar divisor of a meromorphic connection consists of isolated points and the Stokes phenomenon describes the behaviour of solutions in various sectorial domains around these points, in dimension  $\geq 2$  the divisor is no more discrete and the sectorial domains extend in some way all along the divisor. Moreover, questions like pull-back and push-forward by holomorphic maps lead to single out the sheaf-theoretic approach to the Stokes phenomenon. Above the usual complex geometry of the underlying complex manifold with its divisor lives a “wild complex geometry” governing the Stokes phenomenon.

One of the sought applications of this sheaf-theoretic approach, named Stokes-perverse sheaf, is to answer a question that S. Bloch asked me some years ago: to define a sheaf-theoretical Fourier transform over  $\mathbb{Q}$  (say) taking into account the Stokes data. Note that the unpublished manuscript [7] gave an answer to this question (see also the recent work [69] of T. Mochizuki). The need of such an extension to dimension bigger than one also shows up in [21, p. 116].<sup>1</sup>

One of the main problems in the “perverse” approach is to understand on which spaces the sheaves are to be defined. In dimension one, Deligne replaces first a Riemann surface by its real oriented blow-up space at the singularities of the meromorphic connection, getting a surface with boundary, and endows the extended

---

<sup>1</sup>Deligne writes: “On aimerait dire (mais ceci nous obligerait à quitter la dimension 1). . .”

local system of horizontal sections of the connection with a “Stokes filtration” on the boundary. This is a filtration indexed by an ordered local system. We propose to regard such objects as sheaves on the étalé space of the ordered local system (using the notion of étalé space as in [26]). In order to obtain a perfect correspondence with holonomic  $\mathcal{D}$ -modules, Deligne fills the boundary with discs together with perverse sheaves on them, corresponding to the formal part of the meromorphic connection. The gluing at the boundary between the Stokes-filtered local system and the perverse sheaf is defined through grading the Stokes filtration.

The road is therefore a priori well paved and the program can be clearly drafted:

1. To define the notion of Stokes-constructible sheaf on a manifold and a  $t$ -structure in its derived category, in order to recover the category of Stokes-perverse sheaves on a complex manifold as the heart of this  $t$ -structure.
2. To exhibit a Riemann–Hilbert correspondence RH between holonomic  $\mathcal{D}$ -modules and Stokes-perverse sheaves and to prove that it is an equivalence of categories.
3. To define the direct image functor in the derived category of Stokes-constructible sheaves and prove the compatibility of RH when taking direct images of holonomic  $\mathcal{D}$ -modules.

An answer to the latter question would give a way to compute Stokes data of the asymptotic behaviour of integrals of multivalued functions which satisfy themselves a holonomic system of differential equations.

While we realize the first two points of the program in dimension one, by making a little more explicit the contents of [17, 19], we do not go to the end in dimension bigger than one, as we only treat the Stokes-perverse counterpart of meromorphic connections, not holonomic  $\mathcal{D}$ -modules. The reason is that some new phenomena appear, which were invisible in dimension one.

In order to make them visible, let us consider a complex manifold  $X$  endowed with a divisor  $D$ . In dimension one, the topological space to be considered is the oriented real blow-up space  $\widetilde{X}$  of  $X$  along  $D$ , and meromorphic connections on  $X$  with poles on  $D$  are in one-to-one correspondence with local systems on  $X \setminus D$  whose extension to  $\widetilde{X}$  is equipped with a Stokes filtration along  $\partial\widetilde{X}$ . If  $\dim X \geq 2$ , in order to remain in the realm of local systems, a simplification of the underlying geometric situation seems unavoidable in general, so that we treat the case of a divisor with normal crossings (with all components smooth), and a generic assumption has also to be made on the connection, called *goodness*. Variants of this genericity condition have occurred in asymptotic analysis (e.g. in [48]) or when considering the extension of the Levelt–Turrittin theorem to many variables (e.g. in [57]). We define the notion of good stratified  $\mathcal{I}$ -covering of  $\partial\widetilde{D}$ . To any good meromorphic connection and to any good Stokes-filtered local system is associated in a natural way such a good stratified  $\mathcal{I}$ -covering, and the categories to be put into Riemann–Hilbert correspondence are those subcategories of objects having an associated stratified  $\mathcal{I}$ -covering contained in a fixed good one.

This approach remains non-intrinsic, that is, while the category of meromorphic connections with poles along an arbitrary divisor is well defined, we are able to define a Stokes-topological counterpart only with the goodness property. This is an obstacle to define intrinsically a category of Stokes-perverse sheaves. This should be overcome together with the fact that such a category should be stable by direct images, as defined in Chap. 1 for pre- $\mathcal{J}$ -filtrations. Nevertheless, this makes it difficult to use this sheaf-theoretic Stokes theory to obtain certain properties of a meromorphic connection when the polar divisor has arbitrary singularities, or when it has normal crossings but the goodness assumption is not fulfilled. For instance, while the perversity of the irregularity sheaf (a result due to Z. Mebkhout) is easy in the good case along a divisor with normal crossings, we do not have an analogous proof without these assumptions.

Therefore, our approach still remains non-complete with respect to the program above, but already gives strong evidence of its feasibility.

Compared to the approach of T. Mochizuki in [67, 70], which is nicely surveyed in [68], we regard a Stokes-filtered object as an abstract “topological” object, while Mochizuki introduces the Stokes filtration as a filtration of a flat vector bundle. In the recent preprint [69], T. Mochizuki has developed the notion of a Betti structure on a holonomic  $\mathcal{D}$ -module and proved many functorial properties. Viewing the Betti structure as living inside a pre-existing object (a holonomic  $\mathcal{D}$ -module) makes it a little easier to analyse its functorial properties, since the functorial properties of holonomic  $\mathcal{D}$ -modules are already understood. On the other hand, this gives a strong evidence of the existence of a category of Stokes-perverse sheaves with good functorial properties.

In Chap. 1, we develop the notion of Stokes filtration in a general framework under the names of pre- $\mathcal{J}$ -filtration and  $\mathcal{J}$ -filtration, with respect to an ordered sheaf of abelian group  $\mathcal{J}$ . The sheaf  $\mathcal{J}$  for the Stokes filtration in dimension one consists of polar parts of multivalued meromorphic functions of one variable, as originally introduced by Deligne. Its étalé space is Hausdorff, which makes the understanding of a filtration simpler with respect to taking the associated graded sheaf. This chapter may be skipped in a first reading or may serve as a reference for various notions considered starting from Chap. 4.

Part one, starting at Chap. 2, is mainly concerned with dimension one, although Chap. 7 anticipates some results in dimension two, according to the footnote on page x.

In Chap. 2, we essentially redo more concretely the same work as in Chap. 1, in the context of Stokes-filtered local systems on a circle. We prove abelianity of the category in Chap. 3, a fact which follows from the Riemann–Hilbert correspondence, but is proved here directly over any base field  $k$ . In doing so, we introduce the level structure, which was a basic tool in the higher dimensional analogue developed by T. Mochizuki [70], and which was previously considered together with the notion of multisummability [3, 45, 61, 94].

In Chaps. 4 and 5, we develop the notion of a Stokes-perverse sheaf, mainly by following P. Deligne [17, 19] and B. Malgrange [55, Chap. IV.3], and prove the Riemann–Hilbert correspondence. We make explicit the behaviour with respect to

duality, at least at the level of Stokes-filtered local systems, and the main tools are explained in Chap. 4.

Chapter 6 gives two analytic applications of the Riemann–Hilbert correspondence in dimension one. Firstly, the Hermitian dual of a holonomic  $\mathcal{D}$ -module (i.e., the conjugate module of the module of distribution solutions of the original one) on a Riemann surface is shown to be holonomic. Secondly, the local structure of distributions solutions of a holonomic system is analysed.

Chapter 7 presents another application, with a hint of the theory in dimension two, by computing the Stokes filtration of the Laplace transform of a regular holonomic  $\mathcal{D}$ -module on the affine line. We introduce the topological Laplace transformation, and we make precise the relation with duality, of  $\mathcal{D}$ -modules on the one hand, Poincaré–Verdier on the other hand, and their relations. For this chapter, we use tools in dimension 2 which are fully developed in the next chapter.

In Part two we start analysing the Stokes filtration in dimension  $\geq 2$ . Chapter 8 defines the real blow-up space along a family of divisors and the relations between various real blow-up spaces. We pay attention to the global existence of these spaces. The basic sheaf on such real blow-up spaces is the sheaf of holomorphic functions with moderate growth along the divisor. It leads to the moderate de Rham complex of a meromorphic connection. We give some examples of such de Rham complexes, showing how non-goodness can produce higher dimensional cohomology sheaves.

Chapter 9 takes up Chaps. 2 and 3 and introduces the goodness assumption. The construction of the sheaf  $\mathcal{I}$  is given with some care, to make it global along the divisor.

The first approach to the Riemann–Hilbert correspondence in dimension  $\geq 2$  is given in Chap. 10, along a smooth divisor. It can be regarded as obtained by putting a (good) parameter in Chap. 5. The main new argument is the local constancy of the *Stokes sheaf* (Stokes matrices can be chosen locally constant with respect to the parameter).

Chapter 11 analyses the formal properties of good meromorphic connections, following T. Mochizuki [70]. In Chap. 12, we give a proof of the analogue in higher dimension of the Hukuhara–Turrittin theorem, which asymptotically lifts a formal decomposition of the connection. We mainly follow T. Mochizuki’s proof, for which a short account has already been given by M. Hien in [29, Appendix]. We then consider the general case of the Riemann–Hilbert correspondence for good meromorphic connections, and we take this opportunity to answer a question of Kashiwara by proving that the Hermitian dual of a holonomic  $\mathcal{D}$ -module is holonomic (see Chap. 6 in dimension one).

In Chap. 13, we address the question of push-forward and we make explicit a calculation of the Stokes filtration of an exponentially twisted Gauss–Manin system (such a system has already been analysed by C. Roucairol [76–78]). However, the method is dependent on the simple geometric situation, so can hardly be extended directly to the general case, a proof of which has been recently given by T. Mochizuki.

Lastly, Chaps. 14 and 15 are concerned with the nearby cycle functor. In Chap. 14, we first recall the definition of the moderate nearby cycle functor for

holonomic  $\mathcal{D}$ -modules via the Kashiwara–Malgrange  $V$ -filtration, and we review the definition of the irregular nearby cycle functor, due to Deligne. We give a new proof of the preservation of holonomy in a local analytic setting (the proof of Deligne [18] concerns the algebraic setting) when the ambient manifold has dimension two.

In Chap. 15, we give a definition of the nearby cycle functor relative to a holomorphic function for a Stokes-filtered local system and compare it with the notion of moderate nearby cycles of a holonomic  $\mathcal{D}$ -module of Chap. 14 through the Riemann–Hilbert correspondence. We restrict our study to the case of a meromorphic connection with poles along a divisor with normal crossings and a holomorphic function whose zero set is equal to this divisor.

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