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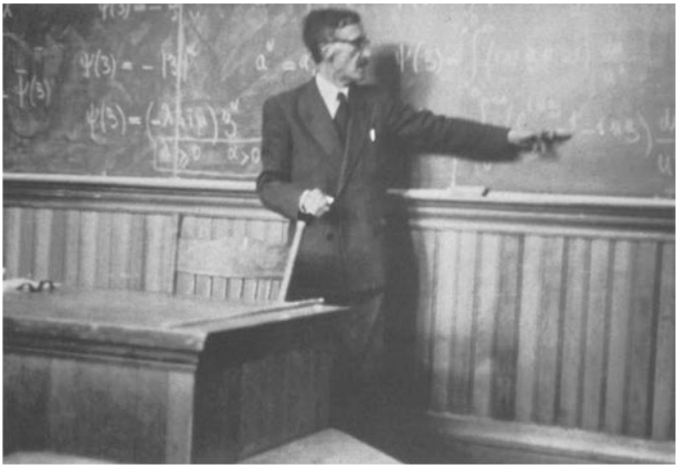

Lévy in Stanford (Permission granted by G.L. Alexanderson)

"Lévy Matters" is a subseries of the Springer Lecture Notes in Mathematics, devoted to the dissemination of important developments in the area of Stochastics that are rooted in the theory of Lévy processes. Each volume will contain state-of-the-art theoretical results as well as applications of this rapidly evolving field, with special emphasis on the case of discontinuous paths. Contributions to this series by leading experts will present or survey new and exciting areas of recent theoretical developments, or will focus on some of the more promising applications in related fields. In this way each volume will constitute a reference text that will serve PhD students, postdoctoral researchers and seasoned researchers alike.

The volumes in this subseries are published under the auspices of the Bernoulli Society.

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Lévy Matters II

Recent Progress in Theory and Applications: Fractional Lévy Fields, and Scale Functions

Springer

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ISBN 978-3-642-31406-3 ISBN 978-3-642-31407-0 (eBook)
DOI 10.1007/978-3-642-31407-0
Springer Heidelberg New York Dordrecht London

Lecture Notes in Mathematics ISSN print edition: 0075-8434
ISSN electronic edition: 1617-9692

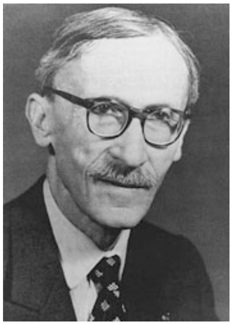
Library of Congress Control Number: 2012945523

Mathematics Subject Classification (2010): Primary: 60G10, 60G40, 60G51, 60G70, 60J10, 60J45
Secondary: 91B28, 91B70, 91B84

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Preface



This second volume of the series "Lévy Matters" consists of two surveys of two topical areas, namely Fractional Lévy Fields by Serge Cohen and the Theory of Scale Functions for Spectrally Negative Lévy Processes by Alexey Kuznetsov, Andreas Kyprianou and Victor Rivero.

Roughly speaking, irregularity is a crucial aspect of random phenomena that appears in many different contexts. An important issue in this direction is to offer tractable mathematical models that encompass the variety of observed behaviours in applications. Fractional Lévy fields are constructed by integration of Lévy random measures; somehow they interpolate between Gaussian and stable random fields. They exhibit a number of interesting features including local asymptotic self-similarity and multi-fractional aspects. Calibration techniques and simulation of fractional Lévy fields constitute important elements for many applications.

A real-valued Lévy process is spectrally negative when it has no positive jumps. In this situation, the distribution of several variables related to the first exit-time from a bounded interval can be specified in terms of the so-called scale functions; the latter also play a fundamental role in other aspects of the theory. Scale functions are characterized by their Laplace transform, but in general no explicit formula is known, and therefore it is crucial in many applications to gather information about their asymptotic behaviour and regularity and to provide efficient numerical methods to compute them.

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From the Preface to Lévy Matters I

Over the past 10–15 years, we have seen a revival of general Lévy processes theory as well as a burst of new applications. In the past, Brownian motion or the Poisson process had been considered as appropriate models for most applications. Nowadays, the need for more realistic modelling of irregular behaviour of phenomena in nature and society like jumps, bursts and extremes has led to a renaissance of the theory of general Lévy processes. Theoretical and applied researchers in fields as diverse as quantum theory, statistical physics, meteorology, seismology, statistics, insurance, finance and telecommunication have realized the enormous flexibility of Lévy models in modelling jumps, tails, dependence and sample path behaviour. Lévy processes or Lévy-driven processes feature slow or rapid structural breaks, extremal behaviour, clustering and clumping of points.

Tools and techniques from related but distinct mathematical fields, such as point processes, stochastic integration, probability theory in abstract spaces and differential geometry, have contributed to a better understanding of Lévy jump processes.

As in many other fields, the enormous power of modern computers has also changed the view of Lévy processes. Simulation methods for paths of Lévy processes and realizations of their functionals have been developed. Monte Carlo simulation makes it possible to determine the distribution of functionals of sample paths of Lévy processes to a high level of accuracy.

This development of Lévy processes was accompanied and triggered by a series of Conferences on Lévy Processes: Theory and Applications. The First and Second Conferences were held in Aarhus (1999, 2002), the Third in Paris (2003), the Fourth in Manchester (2005) and the Fifth in Copenhagen (2007).

To show the broad spectrum of these conferences, the following topics are taken from the announcement of the Copenhagen conference:

- Structural results for Lévy processes: distribution and path properties
- Lévy trees, superprocesses and branching theory
- Fractal processes and fractal phenomena
- Stable and infinitely divisible processes and distributions

- Applications in finance, physics, biosciences and telecommunications
- Lévy processes on abstract structures
- Statistical, numerical and simulation aspects of Lévy processes
- Lévy and stable random fields

At the Conference on Lévy Processes: Theory and Applications in Copenhagen the idea was born to start a series of Lecture Notes on Lévy processes to bear witness of the exciting recent advances in the area of Lévy processes and their applications. Its goal is the dissemination of important developments in theory and applications. Each volume will describe state-of-the-art results of this rapidly evolving subject with special emphasis on the non-Brownian world. Leading experts will present new exciting fields, or surveys of recent developments, or focus on some of the most promising applications. Despite its special character, each article is written in an expository style, normally with an extensive bibliography at the end. In this way each article makes an invaluable comprehensive reference text. The intended audience are PhD and postdoctoral students, or researchers, who want to learn about recent advances in the theory of Lévy processes and to get an overview of new applications in different fields.

Now, with the field in full flourish and with future interest definitely increasing it seemed reasonable to start a series of Lecture Notes in this area, whose individual volumes will appear over time under the common name "Lévy Matters," in tune with the developments in the field. "Lévy Matters" appears as a subseries of the Springer Lecture Notes in Mathematics, thus ensuring wide dissemination of the scientific material. The mainly expository articles should reflect the broadness of the area of Lévy processes.

We take the possibility to acknowledge the very positive collaboration with the relevant Springer staff and the editors of the LN series and the (anonymous) referees of the articles.

We hope that the readers of "Lévy Matters" enjoy learning about the high potential of Lévy processes in theory and applications. Researchers with ideas for contributions to further volumes in the Lévy Matters series are invited to contact any of the editors with proposals or suggestions.

Aarhus, Denmark — Ole E. Barndorff-Nielsen
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Paris, France — Jean Jacod
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Contents

A Short Biography of Paul Lévy

A volume of the series "Lévy Matters" would not be complete without a short sketch about the life and mathematical achievements of the mathematician whose name has been borrowed and used here. This is more a form of tribute to Paul Lévy, who not only invented what we call now Lévy processes, but also is in a sense the founder of the way we are now looking at stochastic processes, with emphasis on the path properties.

Paul Lévy was born in 1886 and lived until 1971. He studied at the Ecole Polytechnique in Paris and was soon appointed as professor of mathematics in the same institution, a position that he held from 1920 to 1959. He started his career as an analyst, with 20 published papers between 1905 (he was then 19 years old) and 1914, and he became interested in probability by chance, so to speak, when asked to give a series of lectures on this topic in 1919 in that same school: this was the starting point of an astounding series of contributions in this field, in parallel with a continuing activity in functional analysis.

Very briefly, one can mention that he is the mathematician who introduced characteristic functions in full generality, proving in particular the characterization theorem and the first "Lévy's theorem" about convergence. This naturally led him to study more deeply the convergence in law with its metric, and also to consider sums of independent variables, a hot topic at the time: Paul Lévy proved a form of the 0-1 law, as well as many other results, for series of independent variables. He also introduced stable and quasi-stable distributions and unravelled their weak and/or strong domains of attractions, simultaneously with Feller.

Then we arrive at the book *Théorie de l'addition des variables aléatoires*, published in 1937, and in which he summaries his findings about what he called "additive processes" (the homogeneous additive processes are now called Lévy processes, but he did not restrict his attention to the homogeneous case). This book contains a host of new ideas and new concepts: the decomposition into the sum of jumps at fixed times and the rest of the process; the Poissonian structure of the jumps for an additive process without fixed times of discontinuities; the "compensation" of those jumps so that one is able to sum up all of them; the fact that the remaining continuous part is Gaussian. As a consequence, he implicitly gave the formula

providing the form of all additive processes without fixed discontinuities, now called the Lévy–Itô formula, and he proved the Lévy–Khintchine formula for the characteristic functions of all infinitely divisible distributions. But, as fundamental as all those results are, this book contains more: new methods, like martingales which, although not given a name, are used in a fundamental way; and also a new way of looking at processes, which is the "pathwise" way: he was certainly the first to understand the importance of looking at and describing the paths of a stochastic process, instead of considering that everything is encapsulated into the distribution of the processes.

This is of course not the end of the story. Paul Lévy undertook a very deep analysis of Brownian motion, culminating in his book *Processus stochastiques et mouvement brownien* in 1948, completed by a second edition in 1965. This is a remarkable achievement, in the spirit of path properties, and again it contains so many deep results: the Lévy modulus of continuity, the Hausdorff dimension of the path, the multiple points, and the Lévy characterization theorem. He introduced local time, and proved the arc-sine law. He was also the first to consider genuine stochastic integrals, with the area formula. In this topic again, his ideas have been the origin of a huge amount of subsequent work, which is still going on. It also laid some of the basis for the fine study of Markov processes, like the local time again, or the new concept of instantaneous state. He also initiated the topic of multi-parameter stochastic processes, introducing in particular the multi-parameter Brownian motion.

As should be quite clear, the account given here does not describe the whole of Paul Lévy's mathematical achievements, and one can consult for many more details the first paper (by Michel Loève) published in the first issue of the *Annals of Probability* (1973). It also does not account for the humanity and gentleness of the person Paul Lévy. But I would like to end this short exposition of Paul Lévy's work by hoping that this series will contribute to fulfilling the program, which he initiated.

Paris, France Jean Jacod