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Prime Divisors and Noncommutative Valuation Theory

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Introduction

The classical theory of valuation rings in fields has natural applications in algebraic geometry as well as in number theory, e.g. as local rings of nonsingular points of curves or, respectively, as prime localizations of rings of integers in number fields. In these applications the rings are Noetherian and the valuations are discrete, that is the value group is the additive group of the integers, $\mathbb{Z}$. The fact that non-discrete valuation rings are not Noetherian may be the reason that general Γ-valuations with totally ordered value group Γ, which need not be discrete, have not been used to the same extent as discrete ones in the aforementioned branches of classical mathematics.

However, function field extensions of arbitrary transcendence degree have been considered; the transcendence degree is recorded as the dimension of some corresponding algebraic variety, but it is also the maximal rank of a valuation ring in the function field. Algebraic geometry easily survives by using the theory of local Noetherian rings, avoiding the non-Noetherian valuation rings dominating them. The algebraic theory in this book deals with generalized Γ-valuations, often related to Γ-filtrations, for totally ordered groups Γ, at places even totally ordered semigroups appear. A source of problems and new ideas is the extension theory for valuations from a subfield to the field. This theory is well developed for number fields in connection with Galois theory, but is perhaps less used for field extensions of finite transcendence degree probably because the (Noetherian) geometric approach is more popular. We want to consider more general algebra extensions including skewfields, matrix rings, etc.

Therefore we develop in Chap. 1 the general theory starting from a noncommutative background, expounding a minimal commutative background when necessary. The primes or prime places introduced provide a very general approach to a valuation theory for associative rings and algebras; this originates in the theory of pseudoplaces as in [70, 71] and the work of Van Geel [68, 69]. The theory is related to the consideration of filtrations and gradations by totally ordered groups; this is the topic of Sect. 1.3 (and Sect. 1.8). Originally pseudoplaces were developed with the aim to apply them to the construction of generic crossed products and generic skewfields, hence it is not really surprising that an interesting class of algebras

inviting the development of a generalized valuation theory is the class of central simple algebras, or more generally finitely dimensional algebras over fields. Thus our theory has direct links to the theory of orders and maximal orders; this will be the subject of Chap. 2. The second part of Chap. 1 is devoted to some particular primes: Dubrovin rings. After the basic theory in Sect. 1.4 and the ideal theory in Sect. 1.5 we focus on Dubrovin valuations of finite dimensional (simple) algebras in Sect. 1.6 and on Gauss extensions in Sect. 1.7. To a generalized valuation ring corresponds a reduction of the original algebra and the relation between properties of the reduced algebra and the original one should be clarified. There are properties of homological type or of (noncommutative) geometric type; in Appendix 1.9 we provide some theory related to Auslander regularity of the algebras under consideration.

Chapter 2 deals with orders. It is not our aim to provide a complete theory of orders in (semisimple) Artinian algebras, but we include a short introduction to this important root of the theory relating to the fundamental work of Auslander, Reiner and others (see [57]).

Arithmetical rings with suitable divisorial properties and noncommutative Krull rings are amongst the natural study objects here (Sects. 2.1 and 2.2). Then we consider Ore extensions over Krull orders and noncommutative valuation rings of Ore extensions of simple Artinian rings in Sects. 2.3 and 2.4. To end the chapter we arrive at an arithmetical divisor theory, a noncommutative divisor calculus, leading to a noncommutative Riemann–Roch theorem over a central curve. The latter result fits in the geometry of rings with polynomial identities, cf. [76]; it also fits in this book because it provides a good example of a concrete, almost calculative, application of noncommutative valuation theory, even if here only discrete valuations appear.

In Chap. 3 we leave the biotope of finite dimensional algebras and turn attention to a recently popular class of infinite dimensional algebras containing quantum groups, quantized algebras and deformations. Usually examples of such algebras are given by generators and relations and therefore they are equipped with a standard filtration. The reduction of such an algebra at a central valuation can be "good" or "bad" depending on how the defining relations reduce modulo the maximal ideal of the central valuation ring. Similar phenomena exist in algebraic geometry related to good reduction of (elliptic) curves but now the reducing "equations" are elements of the ideal of relations in the free algebra. The property of good reduction allows the extension of the central valuation to the (skewfield of fractions of the) noncommutative algebra; this is the topic of Sect. 3.1. To establish a case where we can completely calculate all valuations we focus on the Weyl field in Sect. 3.2 and provide some divisor theory related to a very peculiar subring in Sect. 3.3.

Finally we consider finite dimensional Hopf algebras in the last section and extend the valuation filtration of a central valuation to a Hopf filtration on the Hopf algebra. The filtration part of degree zero is a Hopf order and we show how certain Hopf orders, generalizing Larson orders in the case of finite group rings, may be explicitly constructed. The explicit examples over number rings exhibit the importance of ramification properties of the value function corresponding to the Hopf order.

In our opinion noncommutative extensions of valuation theory deserve to be further integrated in noncommutative algebra (and noncommutative geometry too). We do realize that this book does not provide a finished picture of possible noncommutative extensions of valuation theory, but we hope to have opened a few doors for possible new developments pointing at interesting directions for further research.

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