Lecture Notes in Mathematics 2057

For further volumes:
http://www.springer.com/series/304

Andrzej Cegielski

Iterative Methods for Fixed Point Problems in Hilbert Spaces

Springer

Andrzej Cegielski
University of Zielona Góra
Faculty of Mathematics,
Computer Science and Econometrics
Zielona Góra, Poland

ISBN 978-3-642-30900-7 ISBN 978-3-642-30901-4 (eBook)
DOI 10.1007/978-3-642-30901-4
Springer Heidelberg New York Dordrecht London

Lecture Notes in Mathematics ISSN print edition: 0075-8434
ISSN electronic edition: 1617-9692

Library of Congress Control Number: 2012945521

Mathematics Subject Classification (2010): 47-02, 49-02, 65-02, 90-02, 47H09, 47J25, 37C25, 65F10, 65K15, 90C25

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

To my family:
Elżbieta, Joanna, Gustaw,
Szymon and Katarzyna

Motto:
If you want to find the source, you have to go up, against the current

[John Paul II]

Preface

In this monograph we deal with iteration methods for finding fixed points (if they exist) of nonexpansive operators defined on a Hilbert space, i.e., operators T having the property

$$d(Tx, Ty) \leq cd(x, y) \text{ for all } x, y \in X \tag{1}$$

and for some constant $c \in [0, 1]$. The origin of these methods dates back to 1920, when Stefan Banach (1892–1945) formulated his famous contraction mapping principle. Banach proved that if $T : X \to X$ is a contraction (an operator satisfying (1) with $c < 1$) defined on a complete metric space, then T has a unique fixed point x^*, i.e., a point for which $Tx^* = x^*$. Furthermore, for any $x \in X$ a sequence $\{T^k x\}_{k=0}^{\infty}$ converges geometrically to x^*. Many practical problems can be reduced to finding a fixed point of a nonexpansive operator or a common fixed point of a family of nonexpansive operators. A simple example is a system of linear equations $Ax = b$. Any solution of this system can be identified with a common fixed point of (nonexpansive) operators of orthogonal projections onto hyperplanes corresponding to particular equations of the system.

Under some additional conditions a nonexpansive operator has a fixed point (see, e.g., the Browder–Göhde–Kirk fixed point theorem), but the Banach theorem does not guarantee the convergence of the sequence $\{T^k x\}_{k=0}^{\infty}$. Therefore, it is of great interest to develop methods for finding fixed points of nonexpansive operators. The first iterative method for solving a linear system was proposed in 1937 by a Polish mathematician, Stefan Kaczmarz (1890–1945), in a very short paper (3 pages) "Angenäherte Auflösung von Systemen linearer Gleichungen" ("Approximate solution of systems of linear equations") published in *Bulletin International de l'Académie Polonaise des Sciences et des Lettres*. A year later, an Italian mathematician, Gianfranco Cimmino (1908–1989), proposed another iterative method for a linear system. He published his result in a paper "Calcolo approssimato per le soluzioni dei sistemi di equazioni lineari" ("Approximate computation of the solutions of linear systems") in a *La Ricerca Scientifica*. As opposed to earlier methods for solving systems of linear equations, both methods are motivated rather by geometrical operations than algebraic ones. The main operation in both methods

is a cyclic (in the Kaczmarz method) or simultaneous (in the Cimmino method) orthogonal projection onto hyperplanes described by particular equations of the linear system. The results of Kaczmarz and Cimmino were disregarded for several decades except few references. A great interest in these results started in the 1970s, when it suddenly turned out that the Kaczmarz and Cimmmino methods can be efficiently applied in computed tomography (CT), because the mathematical model of CT can be reduced to a solution of large systems of linear equations with sparse and unstructured matrices, and the methods behave well for such systems. At the end of the twentieth century the interest in the Kaczmarz and Cimmino results was still increasing. Both results have become fundamental to modern iterative methods for fixed point problems for nonexpansive operators.

The third result which influenced the development of this area concerns alternative projections onto two subspaces of a Hilbert space. The result was published in 1950 by John von Neumann in a paper "Functional Operators—Vol. II. The Geometry of Orthogonal Spaces" in *Annals of Mathematical Studies*. Von Neumann proved that a sequence generated by his method converges to the projection of the starting point onto the intersection of the subspaces. The results of Kaczmarz, Cimmino, and von Neumann have been generalized several times in the last decades. Today it is known that the convergence holds for an essentially wider class of operators than orthogonal projections onto hyperplanes, e.g., for nonexpansive operators or for quasi-nonexpansive operators, satisfying some additional conditions. All three methods belong today to classical iterative methods for finding fixed points of nonexpansive operators defined on a Hilbert space. These methods served as the basis for several methods, e.g., for: the Landweber method, projected Landweber method, Douglas–Rachford method, sequential projection methods, methods of cyclic and simultaneous subgradient projections, Dos Santos method of extrapolated simultaneous subgradient projections, reflection-projection method, surrogate projection method, and many others. Irrespective of their theoretical value, they have found application in many areas of mathematics, physics, and technology. The most spectacular application of the methods is an intensity-modulated radiation therapy (IMRT).

Iterative methods for finding fixed points of nonexpansive operators in Hilbert spaces have been described in many publications. In this monograph we try to present the methods in a consolidated way. We introduce some classes of operators, give their properties, define iterative methods generated by operators from these classes, and present general convergence theorems. On this basis we present the conditions under which particular methods converge. A large part of the results presented in this monograph can be found in various forms in the literature. We tried, however, to show that the convergence of a big class of iteration methods follows from general properties of some classes of operators and from some general convergence theorems, in particular from Opial's theorem or from its modifications. This theorem was presented in 1967 by a Polish mathematician, Zdzisław Opial (1930–1974), in a paper "Weak convergence of the sequence of successive approximations for nonexpansive mappings" published in the *Bulletin of the American Mathematical Society*.

In this monograph we work with operators defined on a real Hilbert space, although a part of the results presented herein holds for wider classes of spaces. The monograph is divided into five chapters. In Chap. 1, we recall basic definitions and facts from linear algebra, functional analysis, and convex analysis which we apply in the further part of the monograph. In Chap. 2, we introduce some classes of algorithmic operators (i.e., operators which generate some algorithms or iterative methods): nonexpansive operators, quasi-nonexpansive operators, relaxed quasi-nonexpansive operators, cutter operators, firmly nonexpansive operators, metric projection, relaxed firmly nonexpansive operators, averaged operators, strongly nonexpansive operators, and generalized relaxations of algorithmic operators. Then we present the properties of these classes, in particular the relationships among these classes and their closedness with respect to some algebraic operations on the operators from these classes. In Chap. 3, we analyze the convergence properties of the sequences generated by the operators introduced in Chap. 2. Opial's theorem, its generalizations, and modifications for sequences generated by a subclass of the class of quasi-nonexpansive operators play the key role here. In Chap. 4, we apply the properties of classes of operators presented in Chap. 2 to constructions of operators used in many iterative methods for fixed point problems. In this chapter we also give the properties of the following: the alternating projection, simultaneous projection, cyclic projection, Landweber operator, projected Landweber operator, and some generalizations and extrapolations of these operators. In Chap. 5, we apply the results presented in the previous chapters in order to show the convergence of sequences generated by many iterative methods for fixed point problems, some of which are known in the literature, but several are new. The notions and facts presented in the book are illustrated with 61 figures. Each chapter is followed by several exercises.

Many persons have looked through successive versions of the monograph. I am deeply grateful for their helpful remarks. In particular, I would like to express my thanks to Prof. Simeon Reich from the Technion (Israel), Prof. Yair Censor from the University of Haifa (Israel), Prof. Diethard Pallaschke from the University of Karlsruhe (Germany), and Prof. Heinz Bauschke from the University of British Columbia in Okanagan (Canada). Their valuable remarks have contributed to substantial improvements in successive versions of the monograph and have consolidated me in my aim to give the monograph its final shape. I am also very grateful to my colleagues from the University of Zielona Góra, who looked through the final version of the monograph and also made some useful remarks: Prof. Michał Kisielewicz, Prof. Krzysztof Przesławski, and Prof. Jerzy Motyl. I would like to express my thanks to my Ph.D. student, Rafał Zalas, for his help in the preparation of figures which illustrate the notions and facts presented herein and to Danuta Michalak for the technical composition of the monograph. Last but not least, I would like to express my deep gratitude to my wife, Elżbieta, for her understanding and assistance during the preparation of the monograph.

Zielona Góra, Poland
April 2012

Andrzej Cegielski

Contents