Lecture Notes in Mathematics 2056

For further volumes:
http://www.springer.com/series/304

Mahmoud H. Annaby • Zeinab S. Mansour

q-Fractional Calculus and Equations

Springer

Mahmoud H. Annaby
Cairo University
Faculty of Science
Department of Mathematics
Giza, Egypt

Zeinab S. Mansour
King Saud University
Faculty of Science
Department of Mathematics
Riyadh, Kingdom of Saudi Arabia

ISBN 978-3-642-30897-0 ISBN 978-3-642-30898-7 (eBook)
DOI 10.1007/978-3-642-30898-7
Springer Heidelberg New York Dordrecht London

Lecture Notes in Mathematics ISSN print edition: 0075-8434
ISSN electronic edition: 1617-9692

Library of Congress Control Number: 2012945520

Mathematics Subject Classification (2010): 33D15, 26A33, 30C15, 39A13, 39A70

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Dedicated to the Martyrs of the Great Egyptian 2011 Revolution

Foreword

When I was a graduate student in the early 1970s I was interested in q-series, special functions and I had some interest in fractional calculus. The prevailing wisdom at the time was that none of these subjects will lead anywhere. It is amazing how much things have changed since then and here I am, 40 years later, writing a Preface to a wonderful monograph on q-fractional calculus and q-difference equations. I worked with Waleed Al-Salam who was enthusiastic about the subject. Earlier he introduced the operators of q-fractional calculus and in the early 1970s he worked with Arun Verma on q-fractional Leibniz rule. They found q-analogues of part of a series of papers by Tom Osler on fractional calculus, whose first paper appeared in 1970.

In my opinion the recent developments in q-difference equations and q-fractional calculus naturally followed the developments in the theory of q-series and orthogonal polynomials by a quite a few mathematicians under the leadership of Richard Askey and George Andrews. By the late 1980s the importance of difference and q-difference equations, as well as the Askey–Wilson operator equations, became apparent and Mahmoud Annaby started studying the old papers by Adams, Birkhoff, and their collaborators. He started working with students and other mathematicians from Cairo on related problems and they were naturally led to work on operators of fractional and q-fractional calculus as well as difference and q-difference equations.

Fractional calculus has a long history and has recently gone through a period of rapid development. In writing any book one has to pick and choose from a wide range of topics. The present monograph covers a selection of topics in a rigorous way. It starts with elementary calculus of q-differences and q-integration. Then presents a study of q-difference equations. The existence and uniqueness theorems are derived using successive approximations, leading to systems of equations with retarded arguments. Regular q-Sturm–Liouville theory is also introduced; Green's function is constructed and the eigenfunction expansion theorem is established. The Abel integral equation is the prime example of a Volterra integral equation which can be solved via the Riemann–Liouville operators of fractional calculus. The authors study fractional q-calculi of many types including Riemann–Liouville; Grünwald–Letnikov; Caputo; Erdélyi–Kober and Weyl, in some detail. One aspect I liked is the rigorous treatment of fractional q-Leibniz rule and its application. This puts on

solid ground many classical results starting from those early papers of F. H. Jackson down to the recent papers of Al-Salam and Verma. It is nice to see these results written and the domains of their validity precisely specified.

In special functions the tradition is that identities are proved but in many cases one side gives analytic continuation of the other side. A case in point is the Pfaff–Kummer transformation

$$ {}_2F_1(a,b,c;z) = (1-z)^{-a}{}_2F_1(a,c-b;c,z/(z-1)). $$

This is an identity if $|z| < 1$ and $|z| < |1-z|$. On the other hand, the right-hand side gives an analytic continuation of the left-hand side if $|z| > 1$ but $|z| < |1-z|$. Many identities in this work can be considered as analytic continuation formulas to domains wider than their domains of validity.

Some integral equations of Volterra and Abel type work as an introductory material for the study of fractional q-calculi.

The investigation of q-fractional difference equations leads to families of q-Mittag-Leffler functions which are defined and their properties are investigated, especially the distribution, asymptotic and reality of their zeros, establishing q-counterparts of Wiman's results. Fractional q-difference equations are studied; existence and uniqueness theorems are given and classes of Cauchy-type problems are completely solved in terms of families of q-Mittag-Leffler functions. Among many q-analogues of classical results and concepts, q-Laplace and Fourier transforms are studied and their applications are investigated

As I said earlier this monograph is too brief to be encyclopedic, so the authors had to restrict their coverage of topics within the subject. For example dual integral and series equations are not covered but the reader can find this in Sneddon [276]. Another topic of interest is that operators of fractional calculus can be used to construct reproducing kernels with known eigenvalues and eigenfunctions. Another important missing topic is the characterization of the ranges of various fractional integral operators. This is related to the theory of multipliers for the Mellin transform. Some references are [259, 260]. Zeinab Mansour kindly informed me of her joint work in progress on q-analogues of these results. I look forward to seeing this work completed.

I am sure the publication of this book will stimulate further research in this area.

Orlando, FL Mourad Ismail

Preface

This book is a rigorous study of q-fractional calculus and q-fractional difference equations. Our study is developed starting from the work of Agarwal [17], Al-Salam [18, 19], and Al-Salam and Verma [20]. In [17], the q-fractional Riemann–Liouville calculus is defined formally and many properties are given as well. In comparison with the celebrated monographs on fractional calculus, for example, the first book on fractional calculus [227] written by Oldham and Spanier, the books of Samko et al. [269], Kilbas et al. [169], Podlubny [234], and Diethelm [82], we can see that the q-fractional theory is far from being a well-established q-counterpart of the existing fractional theory. However, we hope that the present work would be a takeoff point to establish a more comprehensive q-fractional theory. We would like to mention that our q-study is based on a q-difference operator and its associated right inverse. This q-difference operator goes back to Euler and may go back to Heine and is reintroduced by Jackson in [158]. Sometimes it is called Euler–Jackson q-difference operator or simply Jackson q-difference operator as we do through the entire book. The main objective of this book is to provide such an overview of the basic theory of fractional q-difference equations, methods of their solutions and applications, taking into account the audience of this book, namely the applied scientists interested in developing the q-theory, investigating and exploring its applications. Applications of the classical fractional calculus appeared in many publications. For example, Sneddon's book [276] on mixed boundary value problems from the mid-1960s included a survey of fractional calculus and how to use it to solve integral equation arising in elasticity. In addition, applications of fractional calculus in mathematical physics, probability, and modeling were introduced in the mid-1970s in [261]. Recently, more applications in classical mechanics, particle physics, diffusion systems, viscoelastic and disordered modern electrical systems, modeling and control are in [267]. Perhaps Leibniz [179] did not expect this number of applications when he sent a letter in 1695 to L'Hôpital asking about the meaning of the derivative of order half. From this point of view, we expect that in the long run, many applications of the fractional q-calculus will appear.

This book consists of nine chapters. Chapter 1 provides some basic definitions and properties of q-analysis as the q-difference operator, the q-integral operator, q-special functions, and q-integral transforms. Chapter 2 is a study of the existence and uniqueness of the solutions of first-order systems of q-difference equations and linear q-difference equations. This chapter also includes some results on zeros of q-trigonometric functions and q-Bessel functions. Chapter 3 includes the basic Sturm–Liouville problem formulated and studied in [30]. It also includes the reformulation introduced in [207] of the q^2-Fourier transform introduced by Rubin in [265, 266]. In Chap. 4, we survey the developments in the fractional q-theory since Al-Salam and Agarwal introduced their generalization to Jackson q-integral and derivatives to fractional orders. It contains the fractional q-calculus associated with Al-Salam and Agarwal fractional q-analogue of the Riemann–Liouville fractional derivatives. Chapter 5 is devoted to other approaches of extending the notion of q-integrals and q-derivatives to fractional orders like q-Caputo fractional derivatives and q-Weyl fractional derivatives. In this chapter we show that a generalization to Grünwald–Letnikov fractional derivatives in the q-settings leads to Al-Salam–Agarwal fractional q-derivatives. We outline the generalization of the Askey–Wilson q-difference operator to fractional orders introduced by Ismail and Rahman in [147]. We conclude this chapter with a generalization of the q-difference operator introduced by Rubin in [265, 266] to fractional orders. In Chap. 6, we give a rigorous proof of Al-Salam–Verma fractional q-Leibniz rule [20] and a generalization of the fractional q-Leibniz rule introduced by Agarwal in [15]. We also introduce a fractional q-Leibniz rule associated with Weyl fractional q-operator. This result is a generalization of the result introduced by Purohit in [248]. At the last section of this chapter, we derive some q-identities using the fractional q-Leibniz formulae represented in this chapter. Chapter 7 is fully devoted to q-Mittag-Leffler functions and their major properties. We explore the Mellin–Barnes contour representations and Hankel contour representations of the two q-analogues of the Mittag-Leffler functions considered in this book. Chapter 8 includes fundamental existence and uniqueness theorems for linear and nonlinear fractional q-difference equations as well as first-order systems of fractional q-difference equations, where the q-derivative is either the Riemann–Liouville fractional q-derivative or Caputo fractional q-derivative. Most of the results of this chapter are a generalization of the results mentioned in Chap. 2. In Chap. 9, the last chapter, we investigate the applications of the q-Laplace, q-Mellin, and q^2-Fourier integral transforms to constructing explicit solutions of certain classes of linear fractional q-difference equations. In the appendix, we include tables of fractional q-derivatives of q-special functions and generalized Rodrigues-type formulae for some q-special functions. The bibliography consists of 302 books and articles, including some recent preprints submitted for publications, up to 2011. However, it cannot be considered as a complete bibliography since this discipline is a fast-growing area. But, on the other hand, we believe that the references of the bibliography and references mentioned therein are enough to get a complete overview of the developments occurred in this subject up to the year 2011.

The authors would like to express their gratitude to their colleagues in the Mathematical Analysis seminar of the department of Mathematics, Faculty of Science, Cairo University for the wonderful environment they created. The authors are also grateful to Alexander von Humboldt foundation for supporting their visit to Mathematisches Seminar, A.C.U Zu Kiel, fall 2006 under the grant 3.4-AGY/1039259, where the first seeds of this book were planted. In this respect, they wish to thank professor Walter Bergweiler for hospitality. Mahmoud Annaby is so grateful to Professor Paul Butzer, RWTH-Aachen for the fruitful discussions held during his visits to Lehrstuhl A für Mathematik, RWTH-Aachen, Germany. Again, Annaby thanks Alexander von Humboldt for supporting his visit, 3.4-8131-KAT/1039259, to the Institute of Mathematics, University of Lübeck, where he started the final revision of the book. Thanks also to Professor Jürgen Prestin. The authors would like to express their gratitude to Professor Mourad Ismail for his constant inspiration, encouragement, and support throughout the whole period of preparing this book project. Particularly, for the fruitful discussions he held with Zeinab Mansour during her visit as a Fulbright Scholar in the academic year 2008/2009 at the university of Central Florida, Florida, USA, and at City University of Hong Kong, Hong Kong, July 2011. Moreover, Zeinab Mansour is greatly thankful and grateful to her husband, Abdallah, for his constant support, encouragement, and understanding; with him, a lot of her dreams came true. She is also thankful to her colleague, Dr. Ahmad El-Guindy, for his constant support and fruitful discussions and to Ms. Zeinab Yussuf for her support in the manuscript revision. Finally, Zeinab Mansour would like to thankfully acknowledge the financial support awarded by King Saud University, Riyadh through grant DSFP/MATH 01 and by NPST Program of King Saud University; project number 10-MAT1293-02.

Mahmoud Annaby
Giza, Egypt

Zeinab Mansour
Riyadh, Saudi Arabia

Contents

Acronyms

Here, we collect a list of symbols that have been used in this book.

Sets

$\mathbb{N}$ $= \{1, 2, 3, \ldots\}$, the sets of natural numbers.

$\mathbb{N}_0$ $= \mathbb{N} \cup \{0\}$.

$\mathbb{Z}^-$ $= \{0, -1, -2, -3, \ldots\}$, the set of all non positive integers.

$\mathbb{R}^+$ $= \{x \in \mathbb{R} \, : \, x > 0\}$.

$\mathbb{R}_{q,+}$ $= \left\{q^k \, : \, k \in \mathbb{Z}\right\}$.

$\mathbb{R}_q$ $= \left\{\pm q^k \, : \, k \in \mathbb{Z}\right\}$.

$\mathbb{R}$ The set of all real numbers.

Function Spaces

$L^p_{q,\eta}(0, a)$; $\eta \in \mathbb{R}$, $p \geq 1$ The space of all functions f defined on $(0, a]$ that satisfy $\int_0^a t^\eta |f(t)|^p \, d_qt < \infty$.

$\mathscr{L}^p_{q,\eta}(0, a)$; $\eta \in \mathbb{R}$, $p \geq 1$ The space of all functions f defined on $(0, a]$ that satisfies $t^\eta f^p(t) \in L^p_{q,\eta}(0, x)$, $0 < x \leq a$. See Definition 1.4.1

$\mathscr{A}C_q[0, a]$ The space of all q-absolutely continuous functions defined on $[0, a]$, see Definition 4.3.1.

$L^2_q\big((0, a) \times (0, a)\big)$ See Definition 1.4.3.

$\mathscr{A}C_q^{(n)}[0, a]$ See Definition 4.3.2.

$\mathscr{S}_{q,\mu}$; $\mu > 0$ The space of all functions f defined on a q^{-1}-geometric set A such that

$$|f(xq^{-n})| = O(q^{\mu n(n+\nu)}) \quad \text{as} \quad n \to \infty.$$

$L^p(\mathbb{R}_{q,+})$ The space of all functions f defined on $\mathbb{R}_{q,+}$ that satisfy

$$\int_0^\infty |f(t)|^p \, d_qt < \infty.$$

$L^p(\mathbb{R}_q)$ The space of all functions f defined on $\mathbb{R}_q$ that satisfy

$$\int_{-\infty}^{\infty} |f(t)|^p \, d_q t < \infty.$$

$\mathscr{L}^p(\Omega)$ where $\Omega \subseteq \mathbb{C}$ is a neighborhood or a deleted neighborhood of zero is the space of all functions defined on Ω such that

$$\int_0^{|z|} |f(t)|^p \, d_q t < \infty \quad \text{for all} \quad z \in \Omega.$$

$C_q^n[a,b]$ The space of all continues functions with continuous q-derivatives up to order $n-1$ on the interval $[a,b]$. See Definition 1.4.4.

Functions

$\lceil \cdot \rceil$ Ceiling function, $\lceil x \rceil := \min\{n \in \mathbb{N}_0 : x \leq n\}$.

$\operatorname{Re} z, \operatorname{Im} z$ Real and imaginary part of the complex number z.

$\Gamma_q(z)$ The q-analogue of the Euler's gamma function.

$B_q(\alpha, \beta)$ The q-analogue of the Euler's beta function.

${}_r\phi_s$ The basic hypergeometric function, see (1.9).

$J_\nu^{(1)}(z;q)$ First Jackson q-Bessel function.

$J_\nu^{(2)}(z;q)$ Second Jackson q-Bessel function.

$J_\nu^{(3)}(z;q)$ Third Jackson q-Bessel function.

$e_q(z),\ E_q(z)$ q-analogues of the exponential function.

$\sin_q z,\ \cos_q z,\ \sinh_q z,\ \cosh_q z$ The q-analogues of the sine, cosine, hyperbolic sine, and hyperbolic cosine associated with $e_q(z)$.

$\operatorname{Sin}_q(z), \operatorname{Cos}_q(z), \operatorname{Sinh}_q(z), \operatorname{Cosh}_q(z)$ The q-analogues of the sine, cosine, hyperbolic sine, and hyperbolic cosine associated with $E_q(z)$.

$\sin(z;q), \cos(z;q)$ q-analogues of the sine and cosine functions, see (2.75)–(2.76).

$\operatorname{Sin}(z;q^2), \operatorname{Cos}(z;q^2)$ q^2-analogues of the sine and cosine functions, see (3.70).

$e(z;q^2)$ $= \operatorname{Cos}(-iz;q^2) + i\,\operatorname{Sin}(-iz;q^2),\ z \in \mathbb{C}$

$e_{\nu,\mu}(z;q),\ E_{\nu,\mu}(z;q)$ q-analogues of the two parameter Mittag-Leffler functions, see (7.3).

${}_r\psi_s(z;q)$ q-analogue of the Fox-Wright function, see Definition 1.8.1.

$\operatorname{Erf}(z;q),\ \operatorname{Erf}_q(z),\ \operatorname{erf}(z;q)$ q-analogues of the error function, see (1.73), (1.75), (1.77).

$\gamma_q(s,x),\ \Gamma_q(s,x)$ q-analogues of the incomplete gamma function, see (1.78).

Operators

D_q Jackson q- divided difference operator.

∂_q The q-divided difference operator introduced by Rubin, see (3.77).

W_q is a q-analogue of the Wronskian determinant, see Definition 2.7.1.

$\mathscr{D}_q$ The Askey–Wilson operator.

${}_qL_s, {}_q\mathscr{L}_s$ The q-analogues of the Laplace integral transforms.

$\mathscr{M}_q$ A q type Mellin integral transform.

$\mathscr{F}_q$ A q^2 type Fourier transform.

D_q^α The Riemann–Liouville fractional q-derivative of order α.

${}^cD_q^\alpha$ The q-Caputo fractional q-derivative of order α.

D_{*q} The fractional Caputo q-difference operator of order α.

$\mathscr{D}_q^{k\alpha}$ The sequential Riemann–Liouville fractional q-derivative of order $k\alpha$, $\alpha > 0$, see (8.74).

${}^c\mathscr{D}_q^{k\alpha}$ The sequential Caputo fractional q-derivative of order $k\alpha$, $0\alpha > 0$.

Other Symbols

$W_{q,\alpha}, |W_{q,\alpha}|$ The q, α Wronskian matrix and the q, α Wronskian determinant associated with the Riemann–Liouville fractional q-derivative, respectively, see Definition 8.6.2.

$W_{q,\alpha}^C, |W_{q,\alpha}^C|$ The q, α Wronskian matrix and the q, α Wronskian determinant associated with the Caputo fractional q-derivative, respectively, see Definition 8.7.2.

$w_{\alpha,\beta}(z;q),\ W_{\alpha,\beta}(z;q)$ q-analogues of the Write functions, see (1.83) and (1.84), respectively.

o Sometimes called little-o or Landau symbol. For example, $f = o(g)$ as $x \to a$ means that $\lim_{x\to a} \frac{f(x)}{g(x)} = 0$.

O Sometimes called big-O or Landau symbol. For example. $f = O(g)$ as $x \to a$ means $f(x)| \leq A|g(x)|$ for some constant A and for all x in a neighborhood of the point a.

$\sim$ For example, $f \sim g$ as $x \to a$ means that $\lim_{x\to a} \frac{f(x)}{g(x)} = 1$.