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Rational Points and Arithmetic of Fundamental Groups

Evidence for the Section Conjecture

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To Antonia, Jaden and Lucie
and to Sabine

Preface

The section conjecture, as stated by Grothendieck [Gr83] in a letter to Faltings in 1983, speculates about a representation of rational points in the realm of anabelian geometry. Every k-rational point of a geometrically connected variety X/k gives rise to a conjugacy class of sections

$$s : \pi_1(\mathrm{Spec}(k)) \to \pi_1(X)$$

of the natural projection $\pi_1(X) \to \pi_1(\mathrm{Spec}(k))$ of étale fundamental groups. The section conjecture suggests that the converse also holds for smooth, projective curves of genus at least 2 over fields k that are finitely generated over $\mathbb{Q}$.

If the section conjecture turns out to be true, then it would shed new Galois theoretic light on the old Diophantine problem of describing rational points.

This volume of Lecture Notes consists of the author's Habilitationsschrift and aims to develop the foundations of the anabelian geometry of sections and to present our knowledge about the section conjecture with a natural bias towards the work of the author. In addition, we discuss the section conjecture over number fields from a local to a global point of view and provide detailed discussions of various analogues of the section conjecture, which might serve as supporting evidence in favour of the conjecture.

Acknowledgments

As this present book builds on several years of work, it is quite natural that over this period I have been influenced by many people and institutions and owe to them a certain debt of gratitude that I will now try to balance.

I have worked at several places: the Mathematische Institut Bonn, the Institute for Advanced Study in Princeton for the academic year 2006/2007, the University of Pennsylvania in Philadelphia in 2007/2008, the Isaac Newton Institute in Cambridge during the summer of 2009, and most recently at the Mathematisches Institut Heidelberg and in particular the MAThematics Center Heidelberg (MATCH).

I thank all these institutes, particularly the people there, I thank for the excellent working conditions and the supportive environment that they provided, for the coffee and for the discussions on mathematics, life, the universe, and all the rest.

I am grateful to Florian Pop for his ongoing interest in my work, for the numerous discussions, and for becoming, beyond an academic teacher, a collaborator and friend. I thank Hiroaki Nakamura and Akio Tamagawa for my several stimulating visits to Okayama and Kyoto. To Kirsten Wickelgren and Gereon Quick, I owe thanks for the enlightening discussions on the point of view of homotopy fixed points towards the section conjecture. My thanks also go to Jochen Gärtner for the discussions on Galois groups of number fields with restricted ramification. I thank David Harari for the pleasant collaboration on the descent obstruction, and I am grateful to Hélène Esnault and Olivier Wittenberg for the wonderful, stimulating, and very intense meetings in Essen, Paris, and Heidelberg, and also for their enthusiastic approach towards mathematics.

There are certainly aspects of the section conjecture beyond what is covered in this book, and also some aspects that were originally intended to be included. But then, among other things, Antonia and Jaden would not sleep in their beloved bunk bed. So, by keeping me busy otherwise, my kids earned the honour of forcing me to focus on finding an end point for this piece of work. I am thankful to Antonia, Jaden, and Lucie for the joy that they are and give every day, and for bringing me coffee and breakfast while these final lines were being written in January 2011.

To my parents, Gisela and Michael Stix, I am extremely grateful for being there whenever a helping hand was needed.

Writing my Habilitationsschrift as a culmination of research conducted over several years while being part of a growing family would have been simply impossible without the warm, enthusiastic, and inexhaustible support of my beloved wife, Sabine. I thank her from the bottom of my heart with wholehearted love and admiration, although there are no words that capture the uncountable gratefulness she deserves.

Heidelberg, Germany Jakob Stix

Contents

Introduction

This volume of Lecture Notes explores the anabelian geometry of the section conjecture that was disclosed by Grothendieck [Gr83] in a letter to Faltings in 1983. On page 6 of the letter the conjecture reads:

> …Es sei $\Gamma(X/K)$ die Menge aller K-wertigen Punkte (also "Schnitte") von X über K, man betrachtet die Abbildung
>
> $$\Gamma(X/K) \to \operatorname{Hom\,ext}_{\pi_1(K)}\big(\pi_1(K), \pi_1(X)\big), \tag{7}$$
>
> wo die zweite Menge also die Menge aller "Spaltungen" der Gruppenerweiterung (3) ist, …, oder vielmehr die Menge der üblichen Konjugationsklassen solcher Spaltungen via Aktion der Gruppe $\pi_1(\overline{X})$. Es ist bekannt, daß (7) injektiv ist, und die Hauptvermutung sagt aus, daß sie bijektiv ist. [siehe unten Berichtigung]

This quote leaves us with a question, a remark and an expectation. The question, besides that the quote is in German, posed as: *what is (3) and what is the map in (7)?*, will be answered when we properly state the conjecture below, while the remark follows after the conjecture has been properly stated. The expectation refers to the *Berichtigung* (an erratum) and will be discussed when it becomes necessary.

The Section Conjecture

Let k be a field, let $\bar{k}$ be a fixed separable closure, and let $\mathrm{Gal}_k = \mathrm{Gal}(\bar{k}/k)$ be the absolute Galois group of k. Let X/k be a geometrically connected variety, let $\overline{X} = X \times_k \bar{k}$ be the base change of X to $\bar{k}$, and let $\bar{x} \in \overline{X}$ be a geometric point. The étale fundamental group $\pi_1(X, \bar{x})$ with base point $\bar{x}$ is an extension, see [SGA1] IX Theorem 6.1,

$$1 \to \pi_1(\overline{X}, \bar{x}) \to \pi_1(X, \bar{x}) \to \mathrm{Gal}_k \to 1, \tag{3}$$

where $\pi_1(\overline{X}, \overline{x})$ is the geometric fundamental group of X with base point $\overline{x}$. In the sequel, we denote the extension (3) by

$$\pi_1(X/k),$$

with the notation indicating that (3) captures the relative π_1-datum of the projection $\mathrm{pr} : X \to \mathrm{Spec}(k)$. The choice of a base point $\bar{x}$ with $\mathrm{pr}_*(\bar{x})$ equivalent to the canonical (due to fixing $\bar{k}$) base point

$$* : \mathrm{Spec}(\bar{k}) \to \mathrm{Spec}(k)$$

of $\mathrm{Spec}(k)$, such that $\mathrm{Gal}_k = \pi_1(\mathrm{Spec}(k), *)$, is silently understood.

To a rational point $a \in X(k)$ the functoriality of π_1 for pointed spaces gives rise to a continuous homomorphism

$$s_a \ : \ \mathrm{Gal}_k \to \pi_1(X, \bar{a}),$$

where $\bar{a}$ is a geometric point above a compatible with $*$. An étale path γ from $\bar{a}$ to $\bar{x}$ on $\overline{X}$ defines an isomorphism

$$\gamma(-)\gamma^{-1} \ : \ \pi_1(X, \bar{a}) \xrightarrow{\sim} \pi_1(X, \bar{x})$$

compatible with the projections pr_* to Gal_k. The composition $\gamma(-)\gamma^{-1} \circ s_a$ thus defines a section of

$$\mathrm{pr}_* \ : \ \pi_1(X, \bar{x}) \to \mathrm{Gal}_k$$

or a splitting of (3). Changing the étale path γ on $\overline{X}$ varies the section over a $\pi_1(\overline{X}, \bar{x})$-conjugacy class of splittings/sections, so that the conjugacy class itself, denoted by $[s_a]$, depends only on the rational point $a \in X(k)$. We denote the set of $\pi_1(\overline{X}, \bar{x})$-conjugacy classes of sections of $\pi_1(X/k)$ by

$$\mathscr{S}_{\pi_1(X/k)} = \{s : \mathrm{Gal}_k \to \pi_1(X, \bar{x}) \ ; \ \mathrm{pr}_* \circ s = \mathrm{id}_{\mathrm{Gal}_k}\} /_{\pi_1(\overline{X}, \bar{x})\text{-conjugacy}}.$$

Changing the base point $\bar{x}$ leads to another description of $\mathscr{S}_{\pi_1(X/k)}$ with a canonical identification between the two descriptions, which moreover satisfies the cocycle relation for composing the identifications between three choices of base points. The set $\mathscr{S}_{\pi_1(X/k)}$ is therefore canonically associated to the variety X/k.

Definition 1. The *(profinite) Kummer map* is the well defined map $a \mapsto [s_a]$

$$\kappa \ : \ X(k) \to \mathscr{S}_{\pi_1(X/k)}$$

denoted by (7) in the quote from [Gr83] above, see Sect. 2.4 and in particular Remark 21 for an origin of the terminology.

The section conjecture of Grothendieck gives a conjectural description of the set of all the sections in an arithmetic situation as follows.

Conjecture 2 (Grothendieck [Gr83]). *Let X be a smooth, projective and geometrically connected curve of genus ≥ 2 over a field k that is finitely generated over $\mathbb{Q}$. Then the profinite Kummer map $a \mapsto [s_a]$ is a bijection from the set of rational points $X(k)$ onto the set $\mathscr{S}_{\pi_1(X/k)}$ of $\pi_1(\overline{X}, \bar{x})$-conjugacy classes of sections of $\pi_1(X/k)$.*

Remark 3. The form of the section conjecture given in Conjecture 2 is one of the standard forms for the conjecture. Actually, the quote above shows that Grothendieck originally stated that injectivity is clear and referred to the surjectivity question as the “Hauptvermutung” (Main Conjecture). For more on the issue of injectivity also in cases beyond the claim of the section conjecture we refer to Chap. 7.

The original form of the section conjecture addressed anabelian curves which are not necessarily projective. The affine case however needs the erratum alluded to above, see Conjecture 6 below for a precise version of the conjecture.

We take the opportunity to distinguish the sections s_a by a name.

Definition 4. A *Diophantine section* is a section $s : \mathrm{Gal}_k \to \pi_1(X, \bar{x})$ of the extension (3) for a geometrically connected variety X/k that is of the form $s = s_a$ for a rational point $a \in X(k)$, more precisely: $s = \gamma(-)\gamma^{-1} \circ s_a$ for an étale path γ on $\overline{X}$.

The Erratum

Conjecture 2 should apply to all anabelian varieties, in particular, at least to hyperbolic curves, the only varieties that are known to definitely show anabelian behaviour.

Definition 5. A *hyperbolic curve* is a smooth, geometrically connected curve U/k with a smooth, projective completion $U \subseteq X$ of genus g such that the complement $Y = X \setminus U$ is étale over $\mathrm{Spec}(k)$ and the Euler characteristic

$$\chi_{\overline{U}} = 2 - 2g - \deg(Y)$$

is negative. We will frequently use the term *curve* to include the property of being geometrically connected.

In short: a hyperbolic curve is a map $U \to \mathrm{Spec}(k)$ with completion $X \to \mathrm{Spec}(k)$ that is a good Artin neighbourhood of $\dim(U) = 1$.

An erratum to Conjecture 2 when *projective* gets replaced by *hyperbolic* becomes necessary because a rational point $y \in Y(k)$ gives rise to non-Diophantine sections, see Chap. 18. In the postscript of [Gr83] we can read the following erratum in the case of anabelian curves which are not necessarily projective.

> ... so läßt sich genau angeben, welches die Spaltungsklassen im zweiten Glied sind, die keinem “endlichen” Punkt also keinem Element des ersten Gliedes entsprechen; ... Es sei
>
> $$\pi_1(K)^0 = \text{Kern von } \pi_1(K) \to \hat{\mathbb{Z}}^* \text{ (der zyklotomische Character)}.$$
>
> Gegeben sei eine Spaltung $\pi_1(K) \to \pi_1(X)$, also $\pi_1(K)$ und deshalb auch $\pi_1(K)^0$ operiert auf $\pi_1(\overline{X})$, der geometrischen Fundamentalgruppe. Die Bedingung ist nun, daß die Fixpunktgruppe dieser Aktion nur aus **1** bestehen soll!

Thus Grothendieck states the section conjecture more precisely as follows.

Conjecture 6 (Grothendieck [Gr83]). *Let k be a field that is finitely generated over $\mathbb{Q}$. For a hyperbolic curve U/k with geometric point $\bar{u} \in U$, the profinite Kummer map induces a bijection*

$$\kappa \ : \ U(k) \to \left\{ s \in \mathscr{S}_{\pi_1(U/k)} \ ; \ \begin{array}{c} \mathbf{1} \textit{ is the only fixed point of } \mathrm{Gal}_k^0 \\ \textit{acting on } \pi_1(\overline{U}, \bar{u}) \textit{ via } s \end{array} \right\}$$

where Gal_k^0 *is the kernel of the cyclotomic character* $\mathrm{Gal}_k \to \hat{\mathbb{Z}}^*$.

We explain in Proposition 104 that Conjecture 6 is well posed because the image of the profinite Kummer map satisfies the fixed point condition. Moreover, in Proposition 256 we infinitesimally generalize work of Nakamura to show that the sections of $\mathscr{S}_{\pi_1(X/k)}$ excluded by the fixed point condition all arise from rational points of the boundary Y. The latter type of sections are called *cuspidal sections*, see Chap. 18 for a proper definition.

The State of the Art in the Section Conjecture

The section conjecture suggests a fascinating way of representing rational points in the realm of anabelian geometry. A genuine arithmetic problem, the Diophantine question of rational solutions to certain polynomials, transforms to a description in the different habitat of profinite groups.

Let us now summarize what is known about the section conjecture prior to these notes or due to the work of the author which built up towards it.

(1) The group theory of cuspidal sections was characterised by Nakamura in a series of works [Na90a, Na90b, Na91], see Theorem 253.

(2) The profinite Kummer map

$$\kappa : X(k) \to \mathscr{S}_{\pi_1(X/k)}$$

is indeed injective in many cases, including those claimed in Conjecture 2 and Conjecture 6. Due to Mochizuki [Mo99] §19, we even know that the pro-p version of κ is injective in many cases. All of this is recalled in Chap. 7.

(3) It took a while until the first examples occurred of hyperbolic curves over algebraic number fields that satisfy the section conjecture. Examples with local obstructions at p-adic places are constructed in [Sx10b], see Sect. 10.1. Subsequently, Harari and Szamuely [HaSz09] with the help of Flynn for numerical data constructed curves of genus 2 over $\mathbb{Q}$ that are counter-examples to the Hasse principle and yet satisfy the section conjecture. One more example of this kind, Schinzel's curve, was shown by Wittenberg [Wi12] to satisfy Conjecture 2 making use of the Brauer–Manin obstruction for sections established in [Sx11], see Chap. 11.

Hain showed in [Ha11b], that the generic curve of genus $g \geq 5$ over a field such that the ℓ-adic cyclotomic character has infinite image does not admit sections and thus also satisfies the section conjecture.

All these examples share the following feature: none of them has a section and therefore also none of them has a rational point, which was known before. There is even no example known of a hyperbolic curve such that the space of sections is finite but non-empty.

Moreover, there is no example known, such that Conjecture 6 holds for the hyperbolic curve X/k together with all finite scalar extensions $X' = X \times_k k'$.

(4) A *weak form of the section conjecture*, Conjecture 100, claims that a projective hyperbolic curve X/k over a field k that is finitely generated over $\mathbb{Q}$ admits a rational point as soon as $\pi_1(X/k)$ splits. It is well known that this *weak form* is far from being weak because it is in fact equivalent to Conjecture 6, see Corollary 101. Consequently, it suffices to treat the case of curves with no rational points, see Corollary 102.

A result on Galois descent in [Sx11], see Sects. 3.3 and 9.4, shows that on the contrary we may as well restrict to the case of curves that have at least one rational point, see Corollary 108.

(5) In the arithmetic case, the space of sections $\mathscr{S}_{\pi_1(X/k)}$ is a profinite set, and in particular compact, see [Sx11]. In Chap. 4 and Sect. 9.1 we describe the topology of $\mathscr{S}_{\pi_1(X/k)}$ in detail.

(6) A series of counter-examples to the pro-p versions of the section conjecture was found by Hoshi [Ho10], see Sect. 14.6, even with an infinite space of sections thus dashing the hope that nilpotent methods alone would solve the conjecture or at least help bounding the set of rational points.

(7) Analogues of the section conjecture have served to prove other anabelian conjectures. Tamagawa's success in anabelian geometry for affine hyperbolic curves [Ta97] relies among other ideas on a group theoretic description of Diophantine sections for hyperbolic curves over finite fields, see Sect. 15.4.

In order to achieve anabelian results for hyperbolic curves over sub-p-adic fields, Mochizuki introduced Hodge–Tate sections in [Mo99] and showed that this property is preserved under open maps between fundamental groups of p-adic curves.

(8) The *real section conjecture*, the analogue of Conjecture 2 with $\mathbb{R}$ as its base field, Theorem 229, is also known due to Mochizuki [Mo03], although it is related and could in fact have been deduced earlier from work of Miller [Mi84], Dwyer, Miller and Neisendorfer [DMN89], Carlsson [Ca91], and Lannes [La92] on Sullivan's conjecture in homotopy theory. Plenty of alternative proofs are now known, for example by the author [Sx10b], Pál [Pa11], Wickelgren [Wg10], Esnault and Wittenberg [EsWi09], see Sect. 16.1 for a survey.

(9) The analogue of the section conjecture for hyperbolic curves over p-adic local fields has been addressed recently by Pop and the author [PoSx11]. The result yields a *valuative section conjecture* that is potentially the final statement, if the section conjecture in the narrow sense is wrong over a p-adic local field, see Sect. 16.2. Instead of a rational point a section would then also potentially originate from a very special Berkovich point or even an adic point.

(10) A birational analogue of the section conjecture was proven by Koenigsmann [Ko05] for curves over p-adic local fields. This has been extended recently to higher transcendence degree in [Sx12a]. A minimalistic pro-p version of Koenigsmann's theorem is due to Pop [Po10].

A recent result put together in final form by Harari and the author building on the work of Koenigsmann and Stoll [St06] establishes the birational section conjecture, Conjecture 260, in a few cases for curves over number fields, see Sect. 18.4. This approach has been extended in [Sx12b], and independently by Hoshi [Ho12], to yield a group theoretic description of Diophantine sections in the birational case for curves over $\mathbb{Q}$. The extra condition imposes a restriction on ramification for certain 2-dimensional representations associated to a section, see Theorem 269.

Esnault and Wittenberg [EsWi10] related the existence of a birational abelian section to the existence of a rational zero cycle of degree 1 under favourable arithmetic assumptions.

(11) In the sense of general theory for sections the following has been achieved. Esnault and Wittenberg [EsWi09] established a theory of the cycle class of a section which was in special cases implicitly used by Parshin [Pa90] and Mochizuki [Mo03], see Chap. 6 for a survey.

An attempt to set up a deformation theory for outer Galois representations was explored by Rastegar [Ra11a, Ra11b].

A Guide Through the Lecture Notes

The present book aims to develop the foundations for the anabelian geometry of sections (Part I) not necessarily limited to sections for curves and to present the state of the art in our knowledge about the section conjecture (Part II and Part IV) with a natural bias towards the work of the author. For example, no effort is taken on working in a pro-$\sum$ context or with truncated versions of the extension $\pi_1(X/k)$, since later, if and when we know the conjecture is true, then we probably see easily what is essentially needed in order to state an optimal result in these directions.

Moreover, we discuss the section conjecture over number fields from a local to global point of view (Part III) and provide detailed discussions of various analogues of the section conjecture (Part IV) which might serve as supporting evidence for the conjecture by showing the arithmetic content hidden in a section of the fundamental group extension.

The focus in this work lies on the section conjecture over *algebraic number fields*[1], since we have a vague idea that the general case over fields that are finitely generated over $\mathbb{Q}$ follows from this case. In order to encourage future research we have included open questions and also cases where only partial results were obtained.

[1] We follow the agreement that a *number field* is an algebraic extension of $\mathbb{Q}$ while an *algebraic number field* is a finite algebraic extension of $\mathbb{Q}$.

A reader with some familiarity with the section conjecture should be able to read the various chapters independently and is encouraged to do so. Let us discuss the content in more detail and in order of appearance in view of new and notable results.

(12) In Sect. 9.5 we examine *going up* and *going down* for the section conjecture with respect to a finite étale map. The discussion relies on results of Chap. 3 on Galois descent and fibres.

(13) Section 10.3 contains some new examples of projective hyperbolic curves X/k without rational points constructed by a principle which works over many base fields k.

(14) Let k be an algebraic number field and let X/k be a geometrically connected variety over k. In Chap. 11, which reports on joint work with Harari in [HaSx12], we reformulate the main result of [HaSx12] in terms of a description of the image of the localisation map

$$\mathrm{res} \; : \; \mathscr{S}_{\pi_1(X/k)} \to \prod_v \mathscr{S}_{\pi_1(X/k)}(k_v)$$

with respect to all places v of k and the corresponding completions k_v. The image consists of all tuples of local sections that survive all finite constant descent obstructions, see Theorem 144.

This point of view is pushed further in Chap. 12 towards at least fragments of a non-abelian Tate–Poitou sequence in Theorem 168.

(15) In Sect. 13.5 we show that each k-rational zero-cycle of degree 1 on a smooth projective variety X/k leads to a section of the abelianized fundamental group extension $\pi_1^{\mathrm{ab}}(X/k)$, see Theorem 192. This result aims beyond curves, because it is only remarkable for a variety X with nontrivial torsion in its geometric Néron–Severi group $\mathrm{NS}_{\overline{X}}$. For such X/k the natural surjection

$$\pi_1^{\mathrm{ab}}(\overline{X}) \twoheadrightarrow \pi_1(\overline{\mathrm{Alb}_X^1})$$

is not an isomorphism, recalled in Proposition 69, where $X \to \mathrm{Alb}_X^1$ is the universal Albanese torsor, see [Wi08].

(16) In Sect. 14.7 we find refinements of the counter-examples for the pro-p version of the section conjecture due to Hoshi. Theorem 218 on the one hand relies on a finer use of the known structure of S-unramified pro-p Galois groups of number fields in the *degenerate case*, see [NSW08] Definition 10.9.3. On the other hand we make use of our discussion of the section conjecture over finite fields, see Chap. 15, and in particular the counting result Theorem 226. As a proof of concept we manage to give an explicit new example, namely the smooth projective curve $C/\mathbb{Q}(\zeta_3)$ given by

$$Y^3 = X(X-1)(X-3)(X-9).$$

The curve has genus 3, an injective pro-3 Kummer map

$$\kappa_3 \; : \; C(\mathbb{Q}(\zeta_3)) \to \mathscr{S}_{\pi_1^{\text{pro-3}}(C/\mathbb{Q}(\zeta_3))}$$

with finite image, and an uncountable space of pro-3 sections. It is amusing to note that the example makes use of the Catalan solution $3^2 - 2^3 = 1$.

(17) In Chap. 15 we investigate the profinite Kummer map in the case of projective hyperbolic curves over a finite field $\mathbb{F}_q$. At first this seems strange as $\mathrm{Gal}_{\mathbb{F}_q} = \hat{\mathbb{Z}}$ is profinite free and thus all extensions $\pi_1(X/\mathbb{F}_q)$ will split. But for example, for abelian varieties over $\mathbb{F}_q$, the analogue of the section conjecture holds as a corollary to a theorem of Lang–Tate. For a projective hyperbolic curve $X/\mathbb{F}_q$ however, Theorem 224 states that the profinite Kummer map is never surjective.

The naive approach to Theorem 224 relies on the estimate (15.5) of the Picard number $\#\,\mathrm{Pic}^0_X(\mathbb{F}_q)$ in terms of the genus g of X and the number of rational points $N = \#X(\mathbb{F}_q)$

$$\#\,\mathrm{Pic}^0_X(\mathbb{F}_q) \geq (q-1)^2 \cdot \frac{1 + q^{g-1} + N(g - 2 + \frac{q^{g-1}-1}{q-1})}{(g+1)(q+1) - N},$$

an infinitesimal improvement over the estimates in [LMD90] Theorem 2. We deduce that

$$\#\,\mathrm{Pic}^0_X(\mathbb{F}_q) > \#X(\mathbb{F}_q)$$

in all but few exceptional cases—later called *space filling curves in their Jacobians*. So then there are sections of $\pi_1(X/\mathbb{F}_q)$ that on the abelian level, as sections of $\pi_1^{\mathrm{ab}}(X/\mathbb{F}_q)$, do not come from a point in $X(\mathbb{F}_q)$. We also wrote a program in SAGE [S$^+$08] to compute (all) exceptional cases, see Sect. 15.2.

In the case of a projective hyperbolic curve, a more refined approach computes the cardinality of the space of pro-ℓ sections by cohomology computations for the associated graded of the descending central series, see Sect. 15.3. It turns out that for bounded order of nilpotency, even taking the product over all $\ell \neq p$, we have only finitely many conjugacy classes of sections, while in the limit for every $\ell \neq p$ the space of pro-ℓ sections is uncountable, see Theorem 226. The result relies on work of Labute [La67] used in a careful study of the associated Lie algebra

$$\mathfrak{g} = \mathrm{Lie}\left(\pi^{\text{pro-}\ell}(\overline{X})\right)$$

and a general result on the space of invariants of a finite abelian group acting on $\mathfrak{g}$ being infinite dimensional, see Proposition 207.

(18) Finite fields again occur in Sect. 17.2 in order to help constructing a field $\mathbb{F}$, an infinite algebraic extension of $\mathbb{F}_p$, with the remarkable property that for every smooth, projective geometrically connected variety $X/\mathbb{F}$ of dimension $\dim(X) > 0$ that injects into an abelian variety, the profinite Kummer map

$$\kappa \;:\; X(\mathbb{F}) \to \mathscr{S}_{\pi_1(X/\mathbb{F})}$$

is injective with dense image with respect to the natural topology of Chap. 4. But unfortunately, the map κ is never surjective, see Theorem 243.