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Coulomb Frames in the Normal Bundle of Surfaces in Euclidean Spaces

Topics from Differential Geometry and Geometric Analysis of Surfaces

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To Emil and Mihaela

Preface

These lecture notes are intended for advanced students and young researchers with interests in the analysis of partial differential equations and differential geometry. We investigate the following problems:

- What are geometrical and analytical characteristics of two-dimensional immersions of disc-type in higher-dimensional Euclidean spaces $\mathbb{R}^n$?
- What can we state about the geometry of orthogonal unit normal frames for such surfaces, as a generalization of the classical concept of unit normal vectors?
- Are there special orthogonal unit normal frames for surfaces which are particularly useful for analytical and geometrical purposes, and how can we construct such frames?

To be more explicit, we have in mind:

- Firstly, to extend treatments on elementary differential geometry of surfaces in $\mathbb{R}^3$, as presented for example in the excellent textbooks of Bär [4], Blaschke and Leichtweiß [12], Klingenberg [80], or Kühnel [84], and to continue treatments, for example, from Brauner [14] or Eschenburg and Jost [44], where selected aspects of surface geometry in Euclidean spaces are already discussed, to an analytical theory of surfaces in Euclidean spaces together with its elements of complex analysis and partial differential equations.
- Secondly, to provide a new approach, as comprehensive as possible, to the construction of orthogonal unit normal frames for surfaces which arise from certain geometric variational problems, so-called *normal Coulomb frames,* together with its elements from the theory of non-linear elliptic systems and modern harmonic analysis.

Our lecture notes contain four chapters which are organized as follows:

- *Chapter 1: Surface geometry*

  We present a comprehensive discussion of the differential geometry of surfaces immersed in Euclidean spaces.

This, in particular, includes the definition of orthogonal unit normal frames for surfaces, one central aspect of our analysis, as well as orthogonal transitions between them.

Furthermore, we derive the differential equations of Gauß and Weingarten as well as the corresponding integrability conditions of Codazzi–Mainardi, Gauß and Ricci. Based on these fundamental identities we introduce important curvature quantities of surfaces, for example the curvature tensor of the normal bundle which plays a particular role in our considerations.

Surface geometry benefits a lot from the theory of generalized analytic functions. To give an idea of what this means we want to conclude the first chapter with proving holomorphy of the so-called Hopf vector which in particular allows us to characterize the zeros of the Gauss curvature of minimal surfaces.

- *Chapter 2: Elliptic systems*

This intermediate chapter begins by introducing the theory of non-linear elliptic systems with quadratic growth in the gradient, and then presents some results concerning curvature estimates and theorems of Bernstein-type for surfaces in Euclidean spaces of arbitrary dimensions.

A famous result of S. Bernstein states that a smooth minimal graph in $\mathbb{R}^3$, defined on the whole plane $\mathbb{R}^2$, must necessarily be a plane. Today we know various strategies to prove this result, and the idea goes back to E. Heinz to establish first a curvature estimate and to deduce Bernstein's result in a second step. However, minimal surfaces with higher codimensions do not share this Bernstein property, as one of our main examples $X(w) = (w, w^2) \in \mathbb{R}^4$ with $w = u + iv$ convincingly shows. It is still a great challenge to find geometrical criteria, preferably in terms of the curvature quantities of the surfaces' normal bundles, which guarantee the validity of Bernstein's theorem.

We must admit that we can only discuss briefly some points where we would wish to employ our tools we develop in this book, but up to now we cannot continue to drive further developments.

- *Chapter 3: Normal Coulomb frames in* $\mathbb{R}^4$

With this chapter we begin our study of constructing normal Coulomb frames for surfaces immersed in Euclidean space $\mathbb{R}^4$.

Normal Coulomb frames are critical for a new functional of total torsion. We present the associated Euler–Lagrange equation and discuss its solution via a Neumann boundary value problem. A proof of the "minimal character" of normal Coulomb frames follows immediately.

Using methods from potential theory and complex analysis we establish various analytical tools to control these special frames. For example, we present two different methods to bound their torsion (connection) coefficients. Methods from the theory of generalized analytic functions will play again an important role.

We conclude the third chapter with a class of minimal graphs for which we can explicitly compute normal Coulomb frames.

- *Chapter 4: Normal Coulomb frames in* $\mathbb{R}^{n+2}$

  Now we consider two-dimensional surfaces immersed in Euclidean spaces $\mathbb{R}^{n+2}$ of arbitrary dimension. The construction of normal Coulomb frames turns out to be more intricate and requires a profound analysis of non-linear elliptic systems in two variables.

  The Euler–Lagrange equations of the functional of total torsion are identified as non-linear elliptic systems with quadratic growth in the gradient, and, more exactly, the non-linearity in the gradient is of so-called *curl-type*, while the Euler–Lagrange equations appear in a *div-curl-form.*

  We discuss the interplay between curvatures of the normal bundles and torsion properties of normal Coulomb frames. It turns out that such frames are free of torsion if and only if the normal bundle is flat.

  Existence of normal Coulomb frames is then established by solving a variational problem in a weak sense using ideas of F. Helein [64]. This, of course, ensures minimality, but we are also interested in classical regularity of our frames. For this purpose we employ deep results of the theory of non-linear elliptic systems of div-curl-type and benefit from the work of many authors: E. Heinz, S. Hildebrandt, F. Helein, F. Müller, S. Müller, T. Rivière, F. Sauvigny, A. Schikorra, E.M. Stein, F. Tomi, H.C. Wente, and many others.

Parallel frames in the normal bundle are often studied in the literature. These are special normal Coulomb frames, namely those with vanishing torsion coefficients, and so they only exist if the normal bundle is flat. In our lecture notes we will mainly consider *non-flat normal bundles* and therefore *nonparallel normal frames.*

Parallel normal frames are widely used in physics, see for example da Costa [30] for a geometric presentation of certain physical problems in quantum mechanics or Burchard and Thomas [19] for an analytical description of the dynamics of Euler's elastic curves. The treatment of such problems in the more general context of nonparallel normal frames is surely desirable but must be left open for the future.

Many fundamental mathematical problems are also left open: How can one construct normal Coulomb frames on surfaces of higher topological type or on higher-dimensional manifolds? Is it possible to combine our results with Helein's construction of tangential Coulomb frames on surfaces from [64]? How can one construct Coulomb frames on manifolds immersed in general Riemannian spaces or Lorentzian spaces? This would surely open the door to applications in general relativity or string theory. The reader is invited to join in the discussion.

Most of the results presented here were obtained in a very fruitful collaboration with Frank Müller from the University of Duisburg-Essen. The reader finds our original approaches in [49, 50] and [51].

I would like to thank the members of Springer for their helpful collaboration, for their support and for their care in preparing this work for print.

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Contents

List of Symbols

Domains of definition

B	Open unit disc
$\overline{B}$	Closed unit disc
∂B	Boundary of the unit disc
$\overline{B}_R$	Closed disc of radius R

Normal vectors

N_σ	Unit normal vector
$\mathfrak{N}$	Orthonormal normal frame (ONF)
$\mathbf{R}$	Orthogonal rotation of orthonormal normal frames

Subspaces

$\mathbb{T}_X(w)$	Tangential space at $w \in B$
$\mathbb{N}_X(w)$	Normal space at $w \in B$
$\mathcal{N}(X)$	Normal bundle

Fundamental forms and Hopf vector

g_{ij}, g^{ij}	First fundamental form
$L_{\sigma,ij}$	Second fundamental form w.r.t. N_σ
$\mathcal{H}$	Hopf vector

Connection coefficients

Γ_{ij}^k	Christoffel symbols
$T_{\sigma,i}^\vartheta$	Torsion coefficients
T_σ^ϑ	Complex-valued torsion vector
$\mathbf{T}_i$	Torsion matrix
$\mathcal{T}$	Grassmann type vector

Curvatures and curvature tensors

κ_g	Geodesic curvature
K	Gauss curvature
H	Scalar mean curvature
$\mathfrak{H}$	Mean curvature vector
R^{ℓ}_{ijk}, R_{nijk}	Riemannian curvature tensor

Normal curvatures

$S^{\vartheta}_{\sigma,ij}$	Curvature tensor of the normal bundle
S	Scalar normal curvature
S^{ω}_{σ}	Normal sectional curvature
$\mathfrak{S}$	Normal curvature vector
$\mathbf{S}_{12}$	Normal curvature matrix

Integral functions

τ, $\tau^{(\sigma\omega)}$	Integral functions
$\mathfrak{T}$	Grassmann-type vector

Parametric functionals

$\mathscr{A}[X]$	Area functional
$\mathscr{T}[\mathfrak{N}]$	Total torsion functional