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Degenerate Nonlinear Diffusion Equations

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Preface

The aim of these notes is to include in a unitary presentation some topics related to the theory of degenerate nonlinear diffusion equations, treated in the mathematical framework of evolution equations with multivalued maximal monotone operators in Hilbert spaces. The problems concern nonlinear parabolic equations involving two cases of degeneracy. More exactly, one case is due to the vanishing of the time derivative coefficient and the other is provided by the vanishing of the diffusion coefficient on subsets of positive measure of the domain.

From the mathematical point of view, the results presented in these notes can be considered as general results in the theory of degenerate nonlinear diffusion equations. However, this work does not seek to present an exhaustive study of degenerate diffusion equations, but rather to emphasize some rigorous and efficient functional methods for approaching these problems.

The main objective is to present various techniques in which a degenerate boundary value problem with initial data can be approached and to introduce relevant solving methods different for each case apart. The work focuses on the theoretical part, but some attention is paid to the link between the abstract formulation and examples concerning applications to boundary value problems which describe real phenomena. Numerical simulations by which the theoretical results are applied to some concrete real-world problems are included with a double scope: for verifying the theory and for illustrating the response given by the theoretical results to the problems arisen in applied sciences.

The material is organized in four chapters, each divided into several sections. The Definitions, results (Theorems, Propositions, Lemmas), and figures are continuously numbered inside a chapter.

The readers are assumed to be familiar with functional analysis, partial differential equations, and some concepts and basic results from the theory of monotone operators. However, the book is self-contained as possible, some specific definitions and results being either introduced at the first place where they are evoked, or indicated by citations. The work addresses to advanced

graduate students in mathematics and engineering sciences, researchers in partial differential equations, applied mathematics and control theory. It can serve as a basis for an advanced course and seminars on applied mathematics for students during the Ph.D. level, and in this respect it is aimed to open to the readers the way toward applications.

The writing of these notes has been developed during the visits of the second author to the Department of Mathematics at the University of Bologna, especially in the periods April–May 2010 when she was a visiting professor, thanks for the financial support of Istituto Nazionale di Alta Matematica "F. Severi"—Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni, Italy, May and November 2011. The work is mainly based on the common results obtained with the first author, and some parts of it completed in 2011 are new and not published in other works.

We would like to thank all the reviewers for having lectured this work with obvious patience and carefulness and for all comments, observations, and suggestions which helped to the text improvement.

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Introduction

Before starting the main body of these notes we would like to explain how the equations we shall study arise from real-world problems. Some particularities of these problems can lead to degenerate equations. They involve various interesting mathematical problems whose study will be concretized in general results which, at their turn, can provide useful information while applied to the originary physical problems.

Throughout the work we are concerned with the study of nonlinear degenerate diffusion problems with the unknown function y, consisting basically in the diffusion equation

$$\frac{\partial(u(t,x)y)}{\partial t} - \Delta\beta^*(y) + \nabla\cdot(a(t,x)G(y)) = f(t,x) \text{ in } Q := (0,T)\times\Omega, \tag{1}$$

with initial data and boundary conditions given for the function $y(t,x)$, or in some cases for $u(t,x)y(t,x)$, where the time t runs in $(0,T)$ with T finite and $x = (x_1,\ldots,x_N) \in \Omega$. The domain Ω is an open bounded subset of $\mathbb{R}^N$, with a sufficiently smooth boundary $\partial\Omega$. The boundary conditions can be of Dirichlet type

$$y(t,x) = 0 \text{ on } \Sigma := (0,T)\times\partial\Omega,$$

of Robin type

$$(a(t,x)G(y) - \nabla\beta^*(y))\cdot\nu = \phi(y) + f_\Gamma(t,x) \text{ on } \Sigma,$$

or of Neumann type if $\phi(y) = 0$. Here, ν is the unit outward normal to Γ. Problem (1) with the initial and boundary conditions can model various diffusion processes in sciences.

As an example we refer to the fluid diffusion in nonhomogeneous partially saturated porous media case in which u accounts for the porosity of the medium, a is a vector characterizing the advection of the fluid through the pores, process also assumed to be influenced by the solution by the means

of $G(y)$, and β^* is a (multivalued) function which will be defined a little later in relation with the coefficient of diffusion. The function f expresses the influence of a source or sink distributed in the flow domain. In this case the function y is the fluid saturation in the medium and $\theta(t,x) = u(t,x)y(t,x)$ denotes the volumetric fluid content, or the medium moisture. Another physical process modeled by (1) may be the propagation of a pollutant of concentration y in a saturated porous medium (with all pores filled with fluid). In this application u describes a process of absorption–desorption, namely the retention of the fluid by the solid matrix and the release (of a part) of it after some time. Heat transfer processes are also modeled by equations of type (1). An equation of a similar type is encountered in the models of soil bioremediation or can be deduced under some assumptions from a Keller–Segel chemotaxis model (see [69]). Equations of form (1) with particular coefficients (possibly vanishing) can also characterize nonlinear population dynamics (see [36]), cell growth (see [64]), imaging processes (see [8,9]) and more generally self-organizing phenomena (see [15]). In population dynamics or medical applications y represents the population density.

Some particular properties of β^* place (1) in specific classes of singular diffusion, as we shall see.

In order to justify the physical relevance of the possibly degenerate diffusion problem (1) we shall explain its settlement by giving an example. This will have the role of making clearer how (1), with singular coefficients possibly degenerating in some cases, is deduced from another model (see (2) below) of a physical process. Thus, equations of the more general type (2) can be studied by reducing them to (1).

Let us consider the following equation

$$C(h)\frac{\partial(u(t,x)h)}{\partial t} - \nabla\cdot(k(h)\nabla h) + \nabla\cdot(a(t,x)g(h)) = f(t,x) \text{ in } Q, \tag{2}$$

with an initial condition

$$h(0,x) = h_0(x) \tag{3}$$

and with boundary conditions (of Dirichlet, Neumann or Robin type) which we do not specify at this time. This equation governs the evolution of a certain physical quantity $h(t,x)$.

For instance, (2) particularized in 3D for $g = k$ and $a(x) = (0,0,1)$ is Richards' equations describing the water infiltration into a soil with the porosity u and the conductivity k (see [84], Chaps. 1 and 2). The function h is the pressure in the soil and C is the water capacity. In a porous medium the unsaturated part is the region (or the whole domain) with the pores only partially filled with water, while the saturated part refers to a region where all pores are completely filled with water. By convention, the pressure h is negative in the unsaturated part and positive in the saturated part.

In (2) we assume that C, k and g are nonnegative nonlinear functions defined on (a subset of) $\mathbb{R}$, u and f are real functions defined on Q, u is nonnegative, and a is a vector whose components a_i are real functions defined on Q. The functions C and u are allowed to vanish on a subset of $\mathbb{R}$, or Q, respectively, inducing a certain degeneracy to (2).

In the example concerning Richards' equation we mention that the water capacity C vanishes when the pressure h becomes positive, i.e., in the saturated domain, and it is positive in the unsaturated domain where $h < 0$.

We notice that in the particular case with C vanishing, the time-space domain Q in which the process evolves splits at some time into two subdomains

$$Q_{ns} = \{(t,x);\ C(h(t,x)) > 0\},\quad Q_s = \{(t,x);\ C(h(t,x)) = 0\}, \tag{4}$$

separated by a surface whose position is modified in time. The flow in these subdomains is described by a parabolic equation (in Q_{ns}) and by an elliptic equation (in Q_s). Therefore, a degeneracy of (2) induced by the vanishing of C may lead to a free boundary problem. It is obvious that if $C(h) > 0$ for all h, then (2) remains parabolic (if $u(t,x) > 0$) and the free boundary problem does not occur. In our example from soil sciences (4) is a situation in which a simultaneous unsaturated and saturated flow can occur. The subset Q_s is called the saturated domain and Q_{ns} is the unsaturated one, physically corresponding to two phases of the infiltration process.

To treat (2) we shall make some transformations to bring it to the form (1), which is more appropriate to the functional treatment we shall apply.

For the moment let us keep $u(t,x)$ positive.

Let us assume $h_m \in \mathbb{R}$, and

$$\begin{aligned} C &: [h_m,\infty) \to [0, C_m],\quad C_m > 0,\\ k &: [h_m,\infty) \to [K_m, K_s],\ K_s > K_m \geq 0,\\ g &: [h_m,\infty) \to [g_m, g_s]. \end{aligned}$$

We shall further refer only to some basic and interesting cases from the mathematical point of view. We consider that the functions C, k and g are single valued, continuous and bounded. More generally they may have discontinuities of first order, at most, i.e., at the points where they are not continuous they have finite lateral limits. In this case some modifications will occur in the model and in its mathematical treatment, without essentially changing the arguments.

We define $C^* : [h_m,\infty) \to [y_m, y_s]$, as

$$C^*(h) = y_m + \int_{h_m}^{h} C(\zeta)d\zeta,\ h \geq h_m, \tag{5}$$

and $K^* : [h_m, \infty) \to [K_m^*, \beta_s^*]$ by

$$K^*(h) := K_m^* + \int_{h_m}^{h} k(\zeta)d\zeta,\ \ h \geq h_m,\ \ K_m^* \geq 0 \tag{6}$$

which are continuous nondecreasing functions, where y_s is a real positive number and $y_m = C^*(h_m)$ is a nonnegative number.

We make the function notation $y = C^*(h)$ and assume first that the medium is unsaturated i.e., C is positive. Then C^* turns out to be monotonically increasing and its inverse is $(C^*)^{-1} : [y_m, y_s] \to [h_m, +\infty)$,

$$h = (C^*)^{-1}(y),\ \ y \in [y_m, y_s]. \tag{7}$$

Replacing h in (2) this becomes

$$\frac{\partial(u(t,x)y)}{\partial t} - \Delta\beta^*(y) + \nabla\cdot(a(t,x)G(y)) + C_1(y)\frac{\partial u}{\partial t}(t,x) = f(t,x) \text{ in } Q, \tag{8}$$

where the composed functions

$$\beta^*(y) := (K^* \circ (C^*)^{-1})(y),\ \ y \in [y_m, y_s], \tag{9}$$

$$G(y) := (g \circ (C^*)^{-1})(y),\ \ y \in [y_m, y_s], \tag{10}$$

$$C_1(y) = C((C^*)^{-1}(y)) \cdot (C^*)^{-1}(y) - y,\ \ y \in [y_m, y_s]$$

occur. Moreover, we still define

$$K(y) := (k \circ (C^*)^{-1})(y),\ \ y \in [y_m, y_s]. \tag{11}$$

Now, we shall separate several cases of interest related to the possible vanishing of C on a subset included in (h_m, ∞). As specified before, in the theory of water infiltration in soils $h_m < 0 = h_s$, $y_m = 0$ and $g = k$. At the value $h_s = 0$ the saturation occurs. The function y is called the *water saturation* in soil and the corresponding value $y_s = C^*(h_s)$ is called the *saturation value.*

(a) *The fast diffusion case.* Let us consider that C is continuous

$$C(h) = 0,\ \ h \in [h_s, \infty),\ \ \ C(h) > 0, h \in [h_m, h_s). \tag{12}$$

Then (2) degenerates on $[h_s, \infty)$ and (5) becomes

$$C^*(h) = \begin{cases} y_m + \int_{h_m}^{h} C(\zeta)d\zeta, & h_m \leq h < h_s, \\ y_s, & h \geq h_s, \end{cases}$$

namely $y = C^*(h)$ remains at the saturation value y_s as well as h is positive. In this case one can compute the inverse of C^* on the interval $[h_m, h_s)$ where it is increasing, while for the interval $h > h_s$ the inverse is no longer a function, but a graph, i.e., at the point $y = y_s$ it has as image the whole interval $[h_s, \infty)$. More exactly we have

$$(C^*)^{-1}(y) := \begin{cases} (C^*)^{-1}(y), & y \in [y_m, y_s), \\ [h_s, +\infty), & y = y_s, \end{cases} \tag{13}$$

and so the function $h = (C^*)^{-1}(y)$ is continuous and monotonically increasing on $[y_m, y_s)$ and so-called *multivalued* at $y = y_s$. Then, by a direct replacement in (6) one obtains a function K^* with two branches

$$K^*(h) := \begin{cases} K_m^* + \int_{h_m}^h k(\zeta)d\zeta, & h \in [h_m, h_s), \\ \beta_s^* + K_s h, & h \geq h_s, \end{cases}$$

which plugged in (9) leads to a multivalued function

$$\beta^*(y) := \begin{cases} (K^* \circ (C^*)^{-1})(y), & y \in [y_m, y_s), \\ [\beta_s^*, +\infty), & y = y_s \end{cases} \tag{14}$$

which has the image $[\beta_s^*, +\infty)$ at $y = y_s$. Here, we took $h_s = 0$ and

$$\beta_s^* := \lim_{y \nearrow y_s} (K^* \circ (C^*)^{-1})(y) > 0. \tag{15}$$

We notice that β^* is continuous on $[y_m, y_s)$ and that $K(y_m) = K_m$, $K(y_s) = K_s$. Since the function $(C^*)^{-1}$ is monotonically increasing on $[y_m, y_s)$ we can calculate $\beta^*(y)$ by changing the variable in the integral (6), by denoting $\zeta = (C^*)^{-1}(\xi)$. In this way we get

$$\beta^*(y) = \begin{cases} K_m^* + \int_{y_m}^y \beta(\xi)d\xi, & y \in [y_m, y_s), \\ [\beta_s^*, +\infty), & y = y_s \end{cases} \tag{16}$$

and now in (8) the sign "=" can be replaced by the sign"$\ni$". In the above expression

$$\beta(y) := \frac{k((C^*)^{-1}(y))}{C((C^*)^{-1}(y))}, \text{ for } y \in [y_m, y_s). \tag{17}$$

The function β defines the *diffusion coefficient* (also called diffusivity for certain physical processes) and under the hypotheses made before it is a nonnegative function, satisfying the blow-up property

$$\lim_{y \nearrow y_s} \beta(y) = +\infty. \tag{18}$$

Concerning detailed computations of the above relations and the various hypotheses made for k which may lead to some specific models we refer to [84], Sect. 2. For a further use we define the ratio

$$\rho := \frac{k(h_m)}{C(h_m)} = \frac{k((C^*)^{-1}(y_m))}{C((C^*)^{-1}(y_m))} \tag{19}$$

which is nonnegative. Let us mention that if C and k are defined on $\mathbb{R}$ (as it happens in some models), i.e., if $h_m = -\infty$, then we replace the above ratio by

$$\rho := \lim_{h \to -\infty} \frac{k(h)}{C(h)}$$

which we assume finite.

We mention that the last term on the left-hand side in (8) is a single-valued function because $C_1(y) = -y$ for $y = y_s$. If u does not depend on t it vanishes.

In our example related to Richards' equation, by defining the functions $y = C^*(h)$, representing the water saturation in pores, β (the water diffusivity) and $G = K$ (the water conductivity function) we have passed from Richards' equation written in terms of pressure to its diffusive form (1) written for the water saturation $y \geq y_m \geq 0$. Generally y_m is taken equal with zero.

In the mathematical treatment we shall need to work with these functions defined up to $-\infty$, so that we extend the β at the left of $y = y_m$ by ρ if $\rho > 0$ and by a continuous positive function if $\rho = 0$. The function K and G are extended by $K(y_m)$ and $G(y_m)$ at the left of $y = y_m$.

To resume, this case is characterized by the functions β and β^* defined on the subset $(-\infty, y_s)$, having singular behaviors at $y = y_s$

$$\lim_{y \nearrow y_s} \beta(y) = +\infty, \quad \beta^*(y_s) \in [\beta_s^*, +\infty). \tag{20}$$

A typical example is

$$\beta(y) = \frac{1}{(y_s - y)^{1-p}} \text{ for } 0 < p < 1.$$

By analogy with the classification given by Aronson in [6] we say that this case defines a *fast diffusion* and models a free boundary process.

Therefore, problem (2), which degenerates due to (12) and has been transformed into (1) with a multivalued function β^* is relevant for a free boundary problem. The fact that a degenerate equation (2) corresponding to a free boundary problem is characterized by an equation with a multivalued operator must not be surprising because the extension of a nonlinear function to a multivalued one by "filling in the jumps" with graphs is common in the theory of nonlinear differential equations with discontinuous coefficients

as well as in that modelling free boundary processes (see [14] for various examples).

Under some other assumptions (regarding mostly the discontinuity of these functions) one can get some other particular properties of the functions β, β^* K and G (e.g., they can be multivalued at some points within $[y_m, y_s)$, see [86]).

(b) *The superdiffusion case.* In this case let us allow β_s^* defined in (15) going to infinity, i.e., assume

$$\lim_{y \nearrow y_s} \beta(y) = +\infty, \quad \lim_{y \nearrow y_s} \beta^*(y) = +\infty. \tag{21}$$

The situation in which both β and β^* blow up at $y = y_s$ corresponds to a singular expression of β,

$$\beta(y) = \frac{1}{(y_s - y)^{1-p}} \text{ for } p \leq 0$$

and was defined in [6] as *very fast diffusion*, or *superdiffusion*.

An interesting example from biology is the instantaneous disappearance (due to an extremely high diffusion coefficient) of a population of locusts when its density reaches a certain critical value, y_s.

(c) *The slow diffusion case.* We assume that

$$C(h) > 0 \text{ for } h \in [h_m, \infty) \tag{22}$$

and

$$\lim_{h \to \infty} C(h) = 0. \tag{23}$$

This implies by (17) that $\beta(y) < \infty$ for $y \in [y_m, +\infty)$ and

$$\lim_{y \to \infty} \beta(y) = +\infty.$$

This case which may be illustrated by

$$\beta(y) = y^p \text{ for } p > 1$$

is a *slow diffusion* and the equation with such a diffusion coefficient is known as the porous media equation, because it generally models the diffusion of a gas in a porous medium. It also models nonlinear heat diffusion. If $p = 1$ we get the classical heat equation. Concerning more general studies on diffusion equations we refer the reader to [96–98]. If

$$\lim_{h \to \infty} C(h) \geq C_s > 0$$

then

$$\lim_{y\to\infty} \beta(y) \le \beta_s = \frac{K_s}{C_s} < +\infty$$

and β^* has a sublinear increasing, $\beta^*(y) \le \beta_s y$.

In the cases (b) and (c) the function β^* turns out to be single valued. Case (c) characterizes a unsaturated flow, only.

To conclude this presentation we give some examples of functions C, β, K which are basic in soil sciences. They are empirical relations established by observations. Let us take the porosity constant and by dividing by it we can rewrite (8) with $u = 1$.

First we refer to the hydraulic model of van Genuchten which proposes the hydraulic functions (in a particular case)

$$K(\widetilde{y}) := \begin{cases} K_s^{0.5}\widetilde{y}[1-(1-\widetilde{y}^{1/m})^m]^2 & \text{if } \widetilde{y} < 1, \\ K_s & \text{if } \widetilde{y} = 1, \end{cases}$$

$$\widetilde{y}(h) := \begin{cases} [1+|\alpha h|^{1/(1-m)}]^{-m} & \text{if } h < 0, \\ \widetilde{y_s} & \text{if } h \ge 0, \end{cases}$$

where $m \in (0,1)$, $\widetilde{y}$ is the dimensionless water saturation defined by

$$\widetilde{y} := \frac{y - y_m}{y_s - y_m}$$

and α is a length scaling factor. Obviously, the dimensionless saturation value $\widetilde{y_s}$ is equal to 1 and $K_s = K(\widetilde{y_s})$. The water capacity is then

$$C(h) := \begin{cases} \frac{m}{1-m}\left\{1+|\alpha h|^{\frac{1}{1-m}}\right\}^{-m-1} |\alpha h|^{\frac{m}{1-m}} \frac{|h|}{h}, & \text{if } h < 0 \\ 0, & \text{if } h \ge 0. \end{cases}$$

The various values of the parameter m correspond to more or less nonlinear behaviours of the soil. For example, we can notice that if m is close to 0, there is a very low rate of variation of $\widetilde{y}(h)$. The point at which C attaints its maximum is very close to the saturation point and the hydraulic conductivity K evolves highly nonlinear. This approaches a slow diffusion case. If m is close to 1, we have that

$$\lim_{h\nearrow 0} C(h) = 0, \quad \lim_{\widetilde{y}\nearrow 1} K'(\widetilde{y}) < +\infty$$

and we can notice a nonlinear variation of $\widetilde{y}(h)$, and a more linear behaviour of the hydraulic conductivity. This leads to a fast diffusion model with a strongly nonlinear transport.

The parametric model of Broadbridge and White (see [33]) introduces as hydraulic functions the diffusion coefficient and the conductivity

$$\beta(\widetilde{y}) = \frac{c(c-1)}{(c-\widetilde{y})^2},\ K(\widetilde{y}) = \frac{(c-1)\widetilde{y}^2}{c-\widetilde{y}}, \tag{24}$$

with the same significance as before for $\widetilde{y}$. Here, the hydraulic nonlinearity of the medium is characterized by the parameter c belonging to $(1, +\infty)$. If $c \to 1$ the medium is strongly nonlinear and if $c \to \infty$ the medium behaves weakly nonlinear. We can define

$$\beta^*(\widetilde{y}) = \begin{cases} \frac{(c-1)\widetilde{y}}{c-\widetilde{y}} \text{ if } \widetilde{y} < 1 \\ [1, \infty) \text{ if } \widetilde{y} = 1 \end{cases} \tag{25}$$

and notice that this corresponds to a slow diffusion for $c >> 1$ and to a fast diffusion when c is approaching the value 1.

These are some examples intended to show that the mathematical properties previously considered for the functions β and K are not formal but they can be retrieved in the usual hydraulic functions in practice. These properties, pretty general, will be transformed into mathematical hypotheses for the boundary value problems discussed in the next chapters.

Without giving an exhaustive information we would like to cite some key achievements in the literature regarding existence and uniqueness studies for solutions to the degenerate equation

$$\frac{\partial(g(h))}{\partial t} - \nabla \cdot (b(\nabla h, g(h))) + f(g(h)) = 0.$$

We refer first to the method of entropy solutions introduced by S.N. Krushkov in [75, 76]. Originally this method was devoted to prove L^1-contraction for entropy solutions for scalar conservation laws, i.e., generalized solutions in the sense of distributions satisfying admissibility conditions similar to those of entropy growth in gas dynamics (see also [23]). H.W. Alt and S. Luckhaus established in [4] various existence, uniqueness and regularity results for initial-boundary value problems for quasilinear systems of the above form and studied variational inequalities of elliptic–parabolic equations applying the results to certain Stefan type problems. J. Carillo (see [37, 38]) applied Krushkov's method to nonlinear equations

$$\frac{\partial(g(h))}{\partial t} - \Delta b(h) + \nabla \cdot \phi(h) = 0$$

with g and b continuous nondecreasing functions and ϕ satisfying rather general conditions. F. Otto (see [92]) proved a L^1-contraction principle and uniqueness of solutions for this type of equation by applying Krushkov's

technique only to the time variable. We also mention the works [65, 66] referring to equations with degenerate diffusion operators.

A degenerate, doubly nonlinear parabolic system, with coefficients depending on time and space too, was treated in [70] by approximating it by a nondegenerate one and proving the convergence of approximate solutions.

In the paper [24] a model of the saturated–unsaturated flow lying on a special definition of the boundary conditions that changes during the phenomenon evolution, has been developed also for a finite value of the diffusivity at saturation (which was implied by the assumption that $C(h) > 0$, $\forall h \geq 0$). Following the technique presented in [50] the model was reduced to systems in class of Stefan-like problems of high-order, see [49].

Another aspect which appears to be very relevant for nonlinear diffusion problems refers to the solution dependence on the choice of nonlinear functions involved in the equation. Such a problem in relation with the fluid flow in a porous medium obeying Richards' equation was recently treated by Borsi et al. in [25]. This article deals with the finite-time stability (in fact the continuity) of solutions to nonlinear scalar parabolic boundary value problems in 1-D with respect to variations of parameter functions and gives extensive details about applications in groundwater flows.

The analysis of the well-posedness of (1) in the very fast diffusion case with u constant was approached in the papers [20] (with Robin boundary conditions) and [81] (with nonhomogeneous Dirichlet boundary conditions) in the framework of evolution equations with m-accretive operators in Hilbert spaces. The latter gives also an example of a nonautonomous problem.

A rigorous existence and uniqueness theory for the fast diffusion case with a free boundary occurrence was begun in [82] for $N = 1$ and developed in [83] for $N = 3$, for $u = 1$. The fast diffusion case with the function k (and consequently β) discontinuous was treated in [86].

As another application, we mention a particular degenerate model characterizing diffusion in partially fissured media which was treated in [94] by extending the existence and uniqueness results proved in [48].

To resume, we have established that (1) characterizing a diffusive flow may be deduced by (2) which can be a degenerate equation (due to the vanishing of $C(h)$) and in this case β blows up and β^* turns out to be multivalued at y_s. In this work we shall associate this case with two situations in which (1) still degenerates due to the hypotheses further presented.

The parabolic–elliptic degenerate case. Let us assume that $u(t, x)$ may vanish on a subset Q_0 of Q of zero measure. This leads to a degeneration of (1) into an elliptic equation on Q_0 and we shall call this case *parabolic–elliptic degenerate*.

The diffusion degenerate case. Let $u(t, x) > 0$ and assume that $k(h_m) = 0$. Then, by (17)

$$\begin{cases} \beta(y) > 0 \text{ for } y > y_m \\ \beta(y) = 0 \text{ for } y = y_m. \end{cases}$$

This implies that the diffusion coefficient $\beta(y)$ vanishes on the subset of Q where $y(t,x) = y_m$ and ρ given by (19) is zero. For convenience we shall call this case *diffusion degenerate*.

If $k(h_m) > 0$ the diffusion coefficient β is positive, i.e., $\rho > 0$, and then we deal with a nondegenerate diffusion case.

We indicate the monograph [62] and the articles [2,3,18,19,51–57,61,63] as a few references concerning the theory of linear equations degenerating into elliptic or hyperbolic ones, examples and identification problems in relation to them.

In this book we treat nonlinear equations degenerating in elliptic ones or degenerating due to the vanishing of diffusion coefficient and focus especially on the fast diffusion case which has a special physical and mathematical relevance being associated with a free boundary problem whose character is concisely expressed by the graph β^*.

The set of hypotheses which will be assumed in each chapter are pretty general and follow from the properties of the empirical functions used in practice and envisage especially singular cases. The techniques used to solve the problems in each chapter can be extended to other more general cases including: equations with discontinuous coefficients (see e.g. [86]), K non Lipschitz, hysteretic behaviour of the nonlinear functions (see e.g. [85]).

Since the monotonicity of the main nonlinear term in the equations corresponds to a natural dissipativity assumption for these classes of problems and has an obvious physical meaning, the methods we use are related to the theory of nonlinear evolution equations with monotone (accretive) operators in Hilbert spaces. They provide rigorous and efficient techniques for approaching well-posedness and other mathematical aspects raised by these problems. In few lines we present a short history of the m-accretivity technique. The main results of the theory of nonlinear maximal monotone operators in Banach spaces are essentially due to Minty (see [79, 80]) and Browder (see [34,35]). Further important contributions are due to Brezis (see [26–28]), Lions [77], Rockafeller [93], mainly in connection with the theory of subdifferential type operators. The general theory of nonlinear m-accretive operators in Banach spaces has been developed in the works of Kato (see [72]) and Crandall and Pazy (see [41, 42]) in connection with the theory of semigroups of nonlinear contractions and nonlinear Cauchy problems in Banach spaces. The existence theory for the Cauchy problem associated with nonlinear m-accretive operators in Banach spaces begins with the pioneering papers of Komura (see [74]) and Kato [71] in Hilbert spaces and was extended in a more general setting by several authors. For a complete approach of these problems we refer the readers to the monographs [10, 14]. At the appropriate places in the text we shall specify the concepts related to the m-accretivity techniques and indicate the main results to which the proofs make appeal.

Chapter 1 is concerned in several sections with the study of the existence and various properties of the solutions to (1) which degenerates in an elliptic

equation. We shall treat this case with a nondegenerate diffusion coefficient. Using techniques from the theory of evolution equations with m-accretive operators in Hilbert spaces results of a higher degree of generality are obtained. Briefly, we shall treat the following problems: existence properties of the solution to (1) in the fast diffusion case, with homogeneous Dirichlet boundary conditions, for u function of x only, vanishing on a subset Ω_0 of Ω with meas$(\Omega) > 0$. In the abstract framework of evolution equations one associates to this case the abstract Cauchy problem

$$\frac{d(My)}{dt}(t) + Ay(t) \ni f(t) \text{ a.e. } t \in (0,T), \tag{26}$$
$$(My)(0) = \theta_0$$

where M is not invertible and A is a multivalued nondegenerate operator. The method consists in an approximation of the operators M and A, the proof of the existence of an approximating solution and a convergence result together with a special construction of a weak solution to (26). Sufficient conditions under which uniqueness follows are established.

The Cauchy problem will be studied further under the hypothesis of a very fast diffusion in which the existence proofs necessitate some different arguments than in the fast diffusion case. They will be detailed in a separate section.

Next, the existence of periodic solutions with f periodic as well as other results concerning the longtime behavior of Cauchy problems with periodic data will be investigated.

In Chap. 2 the problems of existence and uniqueness for the Cauchy problem will be approached in the diffusion degenerate case in which (1) with u constant is accompanied by certain Robin boundary conditions. We shall deal with the abstract problem

$$\frac{dy}{dt}(t) + Ay(t) \ni f(t) \text{ a.e. } t \in (0,T), \tag{27}$$
$$y(0) = y_0$$

where A is a multivalued operator and this case will be approached via a time-difference scheme whose stability and convergence will be studied. The existence of periodic solutions as well as the longtime behavior of the solutions to Cauchy problems with periodic data will be further studied in the diffusion degenerate case, too.

In Chap. 3 a nonautonomous parabolic–elliptic degenerate problem with u depending on t and x is studied. This involves the application of some results of Kato and Crandall and Pazy for semigroups generated by time-dependent nonlinear multivalued operators.

Chapter 4 is concerned with an identification problem, approached as a control problem, for the vanishing coefficient u which will be determined from available observations upon the solution. We resort to an auxiliary approximating problem and determine the optimality conditions for an approximating optimal pair. Then we prove a convergence result of the sequence of approximating control pairs (control and state) to a solution to the original problem.

Some sections are concluded by numerical simulations intended to put into evidence the specific features of the solutions in the context of their physical relevance.

The problems we deal in the book involve mainly nonlinear operators, so that for all concepts, definitions and results related to the nonlinear operators in Banach spaces and semigroup theory we refer the readers to the monographs [10–14, 22, 29, 30, 95, 99].