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Fatiha Alabau-Boussouira • Roger Brockett
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Control of Partial Differential Equations

Cetraro, Italy 2010

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Preface

One of the findings of the 1988 Report by the Panel on Future Directions in Control Theory, chaired by Wendell H. Fleming, was:

Many fundamental theoretical issues, such as control of nonlinear multivariable systems, or *control of nonlinear partial differential equations*, are not yet understood.

Nowadays, more than 20 years later, we believe we can say that a lot of fundamental issues concerning the latter topic have definitely been understood, thanks to the efforts of many researchers who produced a large body of results and techniques. And yet, this process has led to an enormous amount of open questions that will need to be addressed by new generations of scientists. Surveying the most important advances of the last two decades and outlining future research directions were the main motivations that led us to organize the CIME Course on Control of Partial Differential Equations that took place in Cetraro (CS, Italy), July 19–23, 2010. We hope this volume, which is one of the outcomes of that event, will provide an ultimate formative step for those who attended the course, and will represent an authoritative reference for those who were unable to do so.

The course consisted of five series of lectures, which are now the source of the chapters of this monograph. Specifically, the following topics were covered:

- *Stabilization of evolution equations* (by Fatiha Alabau-Boussouira): these lectures discussed recent advances, as well as classical methods, for the stabilization of wave-like equations. Special attention was paid to nonlinear problems, memory-damping, and indirect stabilization of coupled PDEs. All the problems were treated by a unified methodology based on energy estimates. It was shown how the introduction of optimal-weight convexity methods leads to easy computable upper energy decay estimates, and how these results can be completed by lower energy estimates for several examples.
- *Control of the Liouville equation* (by Roger Brockett): these equations describe the evolution of an initial density of points that move according to a given differential equation, and may depend on a control which can be chosen in order to satisfy some prescribed goals. This framework also allows to overcome

limitations of the classical theory: for example, the expense required to implement control laws. Several results (e.g., on ensemble control: controlling, with a single control, a finite but often large number of copies of a given system) as well as open problems were presented.

- *Control in fluid mechanics* (by Olivier Glass): the lectures treated various issues related to the controllability of two well-known equations in fluid mechanics, namely the Euler equation for perfect incompressible fluids in both Eulerian and Lagrangian coordinates, and the one-dimensional isentropic Euler equation for compressible fluids in the framework of entropy solutions. Special emphasis was put on the aspects of the theory that are connected with the nonlinear nature of the problem: linearization around an equilibrium gives here no information on the controllability of the nonlinear system.
- *Carleman estimates for elliptic and parabolic equations, with application to control* (by Jérôme Le Rousseau): these are weighted energy estimates for solutions of partial differential equations with weights of exponential type. The lectures derived Carleman estimates for elliptic and parabolic operators using several methods: a microlocal approach where the main tool is the Gårding inequality, and a more computational direct approach. It was also shown how Carleman estimates can be used to provide unique continuation properties, as well as approximate and null controllability results.
- *Control and numerics for the wave equation* (by Enrique Zuazua): these lectures provided a self-contained presentation of the theory that has been developed recently for the numerical analysis of the controllability properties of wave propagation phenomena. The methodology adopted the so-called discrete approach, which consists in analyzing whether the semidiscrete or fully discrete dynamics arising when discretizing the wave equation by means of the most classical scheme of numerical analysis share the property of being controllable, uniformly with respect to the mesh-size parameters, and the corresponding controls converge to the continuous ones as the mesh size tends to zero. All the results were illustrated by means of several numerical experiments.

Besides the above lectures, there were three seminars, given by Karine Beauchard (Some controllability results for the 2D Kolmogorov equation), Sylvain Ervedoza (Regularity of HUM controls for conservative systems and convergence rates for discrete controls), and Lionel Rosier (Control of some dispersive equations for water waves). There were also four communications given by Ido Bright (Periodic optimization suffices for infinite horizon planar optimal control), Khai Tien Nguyen (The regularity of the minimum time function via nonsmooth analysis and geometric measure theory), Camille Laurent (On stabilization and control for the critical Klein-Gordon equation on a 3-D compact manifold), and Vincent Perrollaz (Exact controllability of entropic solutions of scalar conservation laws with three controls). Seminars and communications are not reproduced in these notes.

One important point, contained in the 1988 Report we mentioned above, is that *advances in the control field are made through a combination of mathematics, modeling, computation, and experimentation*. Hoping the reader will find the present

exposition in accord with such a basic principle, we wish to thank the lecturers and authors who designed their contributions in a detailed-yet-focussed form, for helping us realize this project. Overall, we are very grateful to all the 57 participants in the CIME course, for their enthusiasm that created a friendly and stimulating atmosphere in Cetraro. Finally, special gratitude is due to the GDRE CONEDP, for providing the essential support that allowed us to receive and accept a large number of applications, and to the C.I.M.E. Foundation, for making this event possible and for its very helpful assistance before and all along the lectures.

Rome and Paris

Piermarco Cannarsa
Jean-Michel Coron

Contents

1	On Some Recent Advances on Stabilization for Hyperbolic Equations	1
	Fatiha Alabau-Boussouira	
1.1	Introduction	2
1.1.1	On Nonlinear and Memory Stabilization	4
1.1.2	On Indirect Stabilization for Coupled Systems	7
1.2	Notation	8
1.3	Strong Stabilization	8
1.3.1	Dafermos' Strong Stabilization Result	9
1.4	Linear Stabilization	11
1.4.1	Introduction	12
1.4.2	Geometrical Aspects	12
1.4.3	Exponential Decay for Linear Feedbacks	18
1.4.4	The Compactness–Uniqueness Method	20
1.5	Nonlinear Stabilization in Finite Dimensions	25
1.5.1	Nonlinear Gronwall Inequalities	25
1.5.2	A Comparison Lemma	33
1.5.3	Energy Decay Rates Characterization: The Scalar Case	36
1.5.4	The Vectorial Case and Semi-discretized PDE's	42
1.5.5	Examples of Feedbacks and Optimality	45
1.6	Polynomial Feedbacks in Infinite Dimensions	46
1.7	The Optimal-Weight Convexity Method	48
1.7.1	Introduction and Scope	48
1.7.2	Dominant Kinetic Energy Estimates	52
1.7.3	Weight Function As an Optimal Unknown	55
1.7.4	Simplification of the Energy Decay Rates	59
1.7.5	Generalization to Optic Geometric Conditions: The Indirect Optimal-Weight Convexity Method	60

1.7.6	Examples of Feedbacks and Sharp Upper Estimates	63
1.7.7	Lower Energy Estimates	65
1.8	Memory Stabilization	78
1.8.1	Introduction and Motivations	79
1.8.2	Exponential and Polynomial Decaying Kernels	80
1.8.3	General Decaying Kernels and Optimality	81
1.9	Indirect Stabilization for Coupled Systems	84
1.9.1	Introduction and Motivations	85
1.9.2	A Nondifferential Integral Inequality	87
1.9.3	The Case of Coercive Couplings	88
1.10	Bibliographical Comments	94
1.11	Open Problems	95
	References	96
2	Notes on the Control of the Liouville Equation	101
	Roger Brockett	
2.1	Introduction	101
2.2	Some Limitations on Optimal Control Theory	102
2.3	Measuring Implementation Cost	103
2.4	Ensemble Control	105
2.5	The Liouville Equation	108
2.6	Comparison with the Fokker Planck Equation	110
2.7	Sample Problems Involving the Liouville Equation	111
2.8	Controllability	113
2.9	Optimization with Implementation Costs	114
2.10	Controlling the Variance	116
2.11	Ensembles, Symmetric Functions and Thermodynamics	122
	References	129
3	Some Questions of Control in Fluid Mechanics	131
	Olivier Glass	
3.1	Introduction	132
3.1.1	Control Systems	132
3.1.2	Examples	132
3.1.3	Examples of Control Problems	134
3.2	Controllability of the Euler Equation	137
3.2.1	The Control Problem	137
3.2.2	Controllability Results	140
3.2.3	Proof of the Exact Controllability	142
3.2.4	References	156
3.3	Approximate Lagrangian Controllability of the Euler Equation	157
3.3.1	The Question of Lagrangian Controllability	157
3.3.2	Ideas of Proof	161
3.3.3	Comments	169
3.4	Controllability of the 1D Isentropic (Compressible)	
	Euler Equation	171
3.4.1	Introduction	171

3.4.2	Basic Facts on Systems of Conservation Laws	173
3.4.3	The Controllability Problem	185
3.4.4	Some References	187
3.4.5	Sketch of Proof	189
3.4.6	Comments	202
	References	203
4	Carleman Estimates and Some Applications to Control Theory	207
	Jérôme Le Rousseau	
4.1	Introduction	207
4.2	Differential and Pseudo-Differential Operators with a Large Parameter	208
4.3	Local Carleman Estimates for Elliptic Operators	211
4.3.1	The Method of A. Fursikov and O. Yu. Imanuvilov	214
4.4	Unique Continuation	217
4.5	Local Carleman Estimates at the Boundary for Elliptic Operators ..	219
4.6	From Local to Global Inequalities: Patching Estimates Together ...	224
4.7	Estimates for Parabolic Operators	228
4.7.1	Local Estimate Away from the Boundary	228
4.7.2	Alternative Derivation	230
4.7.3	Local Carleman Estimates at the Boundary	233
4.7.4	Global Estimates for Parabolic Operators	235
4.8	Controllability Results for Parabolic Equations	236
4.8.1	Unique Continuation and Applications to Approximate Controllability	237
4.8.2	Null Controllability for the Heat Equation	237
	Appendix: Proofs of Intermediate Results	239
A.1	Proof of the Gårding Inequality	239
A.2	Example of Functions Fulfilling the Sub-ellipticity Condition: Proof of Lemma 4.3.2	240
A.3	Proof of Lemma 4.3.3	241
A.4	Proof of Lemma 4.3.7	242
A.5	Proof of Lemma 4.3.10	242
	References	243
5	The Wave Equation: Control and Numerics	245
	Sylvain Ervedoza and Enrique Zuazua	
5.1	Introduction	246
5.2	Control and Observation of Finite-Dimensional and Abstract Systems	250
5.2.1	Control of Finite-Dimensional Systems	250
5.2.2	Controllability and Observability for Abstract Conservative Systems	255
5.2.3	Smoothness Results for HUM Controls	257
5.3	The Constant Coefficient Wave Equation	263
5.3.1	Problem Formulation: The 1-d Case	263
5.3.2	Observability for the 1-d Wave Equation	267

5.3.3	Computing the Boundary Control	269
5.3.4	The Multidimensional Wave Equation	271
5.3.5	Smoothness Properties	275
5.4	1-d Finite Difference Semidiscretizations	279
5.4.1	Orientation	279
5.4.2	Finite Difference Approximations	280
5.4.3	Nonuniform Observability	282
5.4.4	Blow up of Discrete Controls	287
5.4.5	Numerical Experiments	291
5.5	Remedies for High-Frequency Pathologies	294
5.5.1	Fourier Filtering	294
5.5.2	A Two-Grid Algorithm	297
5.5.3	Tychonoff Regularization	298
5.5.4	Space Semidiscretizations of the 2D Wave Equations	299
5.5.5	A More General Result	302
5.6	Convergence Results	305
5.6.1	A General Procedure for the Convergence of the Discrete Controls	305
5.6.2	Controllability Results	313
5.6.3	Numerical Experiments	323
5.7	Further Comments and Open Problems	326
5.7.1	Further Comments	326
5.7.2	Open Problems	330
	References	334
	List of Participants	341

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