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Almost Periodic Solutions of Impulsive Differential Equations

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To my wife, Ivanka, and our sons,
Trayan and Alex, for their support and
encouragement

Preface

Impulsive differential equations are suitable for the mathematical simulation of evolutionary processes in which the parameters undergo relatively long periods of smooth variation followed by a short-term rapid changes (i.e., jumps) in their values. Processes of this type are often investigated in various fields of science and technology.

The question of the existence and uniqueness of almost periodic solutions of differential equations is an age-old problem of great importance. The concept of almost periodicity was introduced by the Danish mathematician Harald Bohr. In his papers during the period 1923–1925, the fundamentals of the theory of almost periodic functions can be found. Nevertheless, almost periodic functions are very much a topic of research in the theory of differential equations. The interplay between the two theories has enriched both. On one hand, it is now well known that certain problems in celestial mechanics have their natural setting in questions about almost periodic solutions. On the other hand, certain problems in differential equations have led to new definitions and results in almost periodic functions theory. Bohr's theory quickly attracted the attention of very famous researchers, among them V.V. Stepanov, S. Bochner, H. Weyl, N. Wiener, A.S. Besicovitch, A. Markoff, J. von. Neumann, etc. Indeed, a bibliography of papers on almost periodic solutions of ordinary differential equations contains over 400 items. It is still a very active area of research.

At the present time, the qualitative theory of impulsive differential equations has developed rapidly in relation to the investigation of various processes which are subject to impacts during their evolution. Many results on the existence and uniqueness of almost periodic solutions of these equations are obtained.

In this book, a systematic exposition of the results related to almost periodic solutions of impulsive differential equations is given and the potential for their application is illustrated.

Some important features of the monograph are as follows:

1. It is the first book that is dedicated to a systematic development of *almost periodic theory* for *impulsive differential equations.*
2. It fills a void by making available a book which describes existing literature and authors results on the *relations* between the *almost periodicity* and *stability* of the solutions.
3. It shows the manifestations of *direct constructive methods*, where one constructs a uniformly convergent series of almost periodic functions for the solution, as well as of *indirect methods* of showing that certain bounded solutions are almost periodic, by demonstrating how these effective techniques can be applied to investigate almost periodic solutions of impulsive differential equations and provides interesting *applications* of many practical problems of diverse interest.

The book consists of four chapters.

Chapter 1 has an introductory character. In this chapter a description of the systems of impulsive differential equations is made and the main results on the fundamental theory are given: conditions for absence of the phenomenon "beating," theorems for existence, uniqueness, continuability of the solutions. The class of piecewise continuous Lyapunov functions, which are an apparatus in the almost periodic theory, is introduced. Some comparison lemmas and auxiliary assertions, which are used in the remaining three chapters, are exposed. The main definitions and properties of almost periodic sequences and almost periodic piecewise continuous functions are considered.

In Chap. 2, some basic existence and uniqueness results for almost periodic solutions of different classes of impulsive differential equations are given. The hyperbolic impulsive differential equations, impulsive integro-differential equations, forced perturbed impulsive differential equations, impulsive differential equations with perturbations in the linear part, dichotomous impulsive differential systems, impulsive differential equations with variable impulsive perturbations, and impulsive abstract differential equations in Banach space are investigated. The relations between the strong stability and almost periodicity of solutions of impulsive differential equations with fixed moments of impulse effect are considered. Many examples are considered to illustrate the feasibility of the results.

Chapter 3 is dedicated to the existence and uniqueness of almost periodic solutions of impulsive differential equations by Lyapunov method. Almost periodic Lyapunov functions are offered. The existence theorems of almost periodic solutions for impulsive ordinary differential equations, impulsive integro-differential equations, impulsive differential equations with time-varying delays, and nonlinear impulsive functional differential equations are stated. By using the concepts of uniformly positive definite matrix functions and Hamilton–Jacobi–Riccati inequalities, the existence theorems for almost periodic solutions of uncertain impulsive dynamical systems are proved.

Finally, in Chap. 4, the applications of the theory of almost periodicity to impulsive biological models, Lotka–Volterra models, and neural networks are presented. The impulses are considered either as means of perturbations or as control.

The book is addressed to a wide audience of professionals such as mathematicians, applied researches, and practitioners.

The author has the pleasure to express his sincere gratitude to Prof. Ivanka Stamova for her valuable comments and suggestions during the preparation of the manuscript. He is also thankful to all his coauthors, the work with whom expanded his experience.

Sliven, Bulgaria *G. T. Stamov*

Contents

Introduction

I. Impulsive Differential Equations

The necessity to study impulsive differential equations is due to the fact that these equations are useful mathematical tools in modeling of many real processes and phenomena studied in optimal control, biology, mechanics, biotechnologies, medicine, electronics, economics, etc.

For instance, impulsive interruptions are observed in mechanics [13, 25, 85], in radio engineering [13], in communication security [79, 80], in Lotka–Volterra models [2–4, 76, 99, 103, 106, 188, 191, 192], in control theory [75, 104, 118, 146], in neural networks [5, 6, 36, 162, 169, 175–177]. Indeed, the states of many evolutionary processes are often subject to instantaneous perturbations and experience abrupt changes at certain moments of time. The duration of these changes is very short and negligible in comparison with the duration of the process considered, and can be thought of as "momentary" changes or as impulses. Systems with short-term perturbations are often naturally described by impulsive differential equations [15, 20, 62, 66, 94, 138].

Owing to its theoretical and practical significance, the theory of impulsive differential equations has undergone a rapid development in the last couple of decades.

The following examples give a more concrete notion of processes that can be described by impulsive differential equations.

Example 1. One of the first mathematical models which incorporate interaction between two species (predator–prey, or herbivore-plant, or parasitoid-host) was proposed by Alfred Lotka [109] and Vito Volterra [184]. The classical "predator–prey" model is based on the following system of two differential equations

$$\begin{cases} \dot{H}(t) = H(t)[r_1 - bP(t)], \\ \dot{P}(t) = P(t)[-r_2 + cH(t)], \end{cases} \tag{A}$$

where $H(t)$ and $P(t)$ represent the population densities of prey and predator at time t, respectively, $t \geq 0$, $r_1 > 0$ is the intrinsic growth rate of the prey, $r_2 > 0$ is the death rate of the predator or consumer, b and c are the interaction constants. More specifically, the constant b is the per-capita rate of the predator predation and the constant c is the product of the predation per-capita rate and the rate of converting the prey into the predator.

The product $p = p(H) = bH$ of b and H is the predator's functional response (response function) of type I, or rate of prey capture as a function of prey abundance.

There have been many studies in literatures that investigate the population dynamics of the type (A) models. However, in the study of the dynamic relationship between species, the effect of some impulsive factors, which exists widely in the real world, has been ignored. For example, the birth of many species is an annual birth pulse or harvesting. Moreover, the human beings have been harvesting or stocking species at some time, then the species is affected by another impulsive type. Also, impulsive reduction of the population density of a given species is possible after its partial destruction by catching or poisoning with chemicals used at some transitory slots in fishing or agriculture. Such factors have a great impact on the population growth. If we incorporate these impulsive factors into the model of population interaction, the model must be governed by an impulsive differential system.

For example, if at the moment $t = t_k$ the population density of the predator is changed, then we can assume that

$$\Delta P(t_k) = P(t_k^+) - P(t_k^-) = g_k P(t_k), \tag{B}$$

where $P(t_k^-) = P(t_k)$ and $P(t_k^+)$ are the population densities of the predator before and after impulsive perturbation, respectively, and $g_k \in \mathbb{R}$ are constants which characterize the magnitude of the impulsive effect at the moment t_k. If $g_k > 0$, then the population density increases and if $g_k < 0$, then the population density decreases at the moment t_k.

Relations (A) and (B) determine the following system of impulsive differential equations

$$\begin{cases} \dot{H}(t) = H(t)[r_1 - bP(t)], \quad t \neq t_k, \\ \dot{P}(t) = P(t)[-r_2 + cH(t)], \quad t \neq t_k, \\ H(t_k^+) = H(t_k), \quad P(t_k^+) = P(t_k) + g_k P(t_k), \end{cases} \tag{C}$$

where t_k are fixed moments of time, $0 < t_1 < t_2 < \ldots$, $\lim\limits_{k\to\infty} t_k = \infty$.

In mathematical ecology the system (C) denotes a model of the dynamics of a predator–prey system, which is subject to impulsive effects at certain moments of time. By means of such models, it is possible to take into account

the possible environmental changes or other exterior effects due to which the population density of the predator is changed momentary.

Example 2. The most important and useful functional response is the Holling type II function of the form

$$p(H) = \frac{CH}{m+H},$$

where $C > 0$ is the maximal growth rate of the predator, and $m > 0$ is the half-saturation constant. Since the function $p(H)$ depends solely on prey density, it is usually called a *prey-dependent* response function. Predator–prey systems with prey-dependent response have been studied extensively and the dynamics of such systems are now very well understood [77, 88, 125, 135, 181, 192].

Recently, the traditional prey-dependent predator–prey models have been challenged, based on the fact that functional and numerical responses over typical ecological timescales ought to depend on the densities of both prey and predators, especially when predators have to search for food (and therefore have to share or compete for food). Such a functional response is called a *ratio-dependent* response function. Based on the Holling type II function, several biologists proposed a ratio-dependent function of the form

$$p(\frac{H}{P}) = \frac{C\frac{H}{P}}{m+\frac{H}{P}} = \frac{CH}{mP+H}$$

and the following ratio-dependent Lotka–Volterra model

$$\begin{cases} \dot{H}(t) = H(t)\Big[r_1 - aH(t) - \dfrac{CP(t)}{mP(t)+H(t)}\Big], \\ \dot{P}(t) = P(t)\Big[-r_2 + \dfrac{KH(t)}{mP(t)+H(t)}\Big], \end{cases} \tag{D}$$

where K is the conversion rate.

If we introduce time delays in model (D), we will obtain a more realistic approach to the understanding of predator–prey dynamics, and it is interesting and important to study the following delayed modified ratio-dependent Lotka–Volterra system

$$\begin{cases} \dot{H}(t) = H(t)\Big[r_1 - a\displaystyle\int_{-\infty}^{t} k(t-u)H(u)du - \dfrac{CP(t-\tau(t))}{mP(t)+H(t)}\Big], \\ \\ \dot{P}(t) = P(t)\Big[-r_2 + \dfrac{KH(t-\tau(t))}{mP(t)+H(t-\tau(t))}\Big], \end{cases} \tag{E}$$

where $k : \mathbb{R}^+ \to \mathbb{R}^+$ is a measurable function, corresponding to a delay kernel or a weighting factor, which says how much emphasis should be given to the size of the prey population at earlier times to determine the present effect on resource availability, $\tau \in C[\mathbb{R}, \mathbb{R}^+]$.

However, the ecological system is often affected by environmental changes and other human activities. In many practical situations, it is often the case that predator or parasites are released at some transitory time slots and harvest or stock of the species is seasonal or occurs in regular pulses. By means of exterior effects we can control population densities of the prey and predator.

If at certain moments of time biotic and anthropogeneous factors act on the two populations "momentary", then the population numbers vary by jumps. In this case we will study Lotka–Volterra models with impulsive perturbations of the type

$$\begin{cases} \dot{H}(t) = H(t)\Big[r_1 - a\displaystyle\int_{-\infty}^{t} k(t-u)H(u)du \\ \qquad\qquad - \dfrac{CP(t-\tau(t))}{mP(t)+H(t)}\Big],\ t \neq t_k, \\ \dot{P}(t) = P(t)\Big[-r_2 + \dfrac{KH(t-\tau(t))}{mP(t)+H(t-\tau(t))}\Big],\ t \neq t_k, \\ H(t_k^+) = (1+h_k)H(t_k), \quad k = 1, 2, \ldots, \\ P(t_k^+) = (1+g_k)P(t_k), \quad k = 1, 2, \ldots, \end{cases} \tag{F}$$

where $h_k,\ g_k \in \mathbb{R}$ and $t_k,\ k = 1, 2, \ldots$ are fixed moments of impulse effects, $0 < t_1 < t_2 < \ldots,\ \lim\limits_{k\to\infty} t_k = \infty$.

By means of the type (F) models it is possible to investigate one of the most important problems of the mathematical ecology - the problem of stability of the ecosystems, and respectively the problem of the optimal control of such systems.

Example 3. Chua and Yang [42, 43] proposed a novel class of information-processing system called Cellular Neural Networks (CNN) in 1988. Like neural networks, it is a large-scale nonlinear analog circuit which processes signals in real time.

The key features of neural networks are asynchronous parallel processing and global interaction of network elements. For the circuit diagram and

connection pattern, implementation the CNN can be referred to [44]. Impressive applications of neural networks have been proposed for various fields such as optimization, linear and nonlinear programming, associative memory, pattern recognition and computer vision [30, 31, 35, 40–44, 72].

The mathematical model of a Hopfield type CNN is described by the following state equations

$$\dot{x}_i(t) = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j\left(x_j(t)\right) + I_i, \tag{G}$$

or by delay differential equations

$$\dot{x}_i(t) = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j\left(x_j(t)\right) + \sum_{j=1}^{n} b_{ij} f_j\left(x_j(t - \tau_j(t))\right) + I_i, \tag{H}$$

where $i = 1, 2, \ldots, n$, n corresponds to the numbers of units in the neural network, $x_i(t)$ corresponds to the state of the ith unit at time t, $f_j(x_j(t))$ denotes the output of the jth unit at time t, a_{ij} denotes the strength of the jth unit on the ith unit at time t, b_{ij} denotes the strength of the jth unit on the ith unit at time $t - \tau_j(t)$, I_i denotes the external bias on the ith unit, $\tau_j(t)$ corresponds to the transmission delay along the axon of the jth unit and satisfies $0 \le \tau_j(t) \le \tau$ ($\tau = const$), c_i represents the rate with which the ith unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs.

On the other hand, the state of CNN is often subject to instantaneous perturbations and experiences abrupt changes at certain instants which may be caused by switching phenomenon, frequency change or other sudden noise, that is, do exhibit impulsive effects.

Therefore, neural network models with impulsive effects should be more accurate in describing the evolutionary process of the systems.

Let at fixed moments t_k the system (G) or (H) be subject to shock effects due to which the state of the ith unit gets momentary changes. The adequate mathematical models in such situation are the following impulsive CNNs

$$\begin{cases} \dot{x}_i(t) = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j\left(x_j(t)\right) + I_i, \quad t \neq t_k,\ t \geq 0, \\ \\ \Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k) = P_{ik}(x_i(t_k)),\ k = 1, 2, \ldots, \end{cases} \tag{I}$$

or the impulsive system with delays

$$\begin{cases} \dot{x}_i(t) = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j\left(x_j(t)\right) \\ \qquad\qquad + \sum_{j=1}^{n} b_{ij} f_j\left(x_j(t-\tau_j(t))\right) + I_i, \quad t \neq t_k,\ t \geq 0, \\ \Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k) = P_{ik}(x_i(t_k)),\ k = 1, 2, \ldots, \end{cases} \tag{J}$$

where t_k, $k = 1, 2, \ldots$ are the moments of impulsive perturbations and satisfy $0 < t_1 < t_2 < ldots$, $\lim_{k\to\infty} t_k = \infty$ and $P_{ik}(x_i(t_k))$ represents the abrupt change of the state $x_i(t)$ at the impulsive moment t_k.

Such a generalization of the CNN notion should enable us to study different types of classical problems as well as to "control" the solvability of the differential equations (without impulses).

In the examples considered the systems of impulsive differential equations are given by means of a system of differential equations and conditions of jumps. A brief description of impulsive systems is given in Chap. 1.

The mathematical investigations of the impulsive ordinary differential equations mark their beginning with the work of Mil'man and Myshkis [119], 1960. In it some general concepts are given about the systems with impulse effect and the first results on stability of such systems solutions are obtained. In recent years the fundamental and qualitative theory of such equations has been extensively studied. A number of results on existence, uniqueness, continuability, stability, boundedness, oscillations, asymptotic properties, etc. were published [14–18, 20, 61, 62, 87, 91–96, 124, 128, 129, 136, 138, 148–151, 153–161, 165, 166, 178, 199]. These results are obtained in the studying of many models which are using in natural and applied sciences [2, 10, 11, 59, 75, 103–105, 121, 130, 131, 152, 163, 164, 167, 168, 170–176, 178].

II. Almost Periodicity

The concept of almost periodicity is with deep historical roots. One of the oldest problems in astronomy was to explain some curious behavior of the moon, sun, and the planets as viewed against the background of the "fixed stars". For the Greek astronomers the problem was made more difficult by the added restriction that the models for the solar system were to use only uniform linear and uniform circular motions. One such solution, sometimes attributed to Hipparchus and appearing in the Almagest of Ptolemy, is the method of epicycles.

Let P be a planet or the moon. The model of motion of P can be written as

$$r_1 e^{i\lambda_1 t} + r_2 e^{i\lambda_2 t},$$

where $r_1,\ \lambda_1$ and $r_2,\ \lambda_2$ are real constants. When applied to the moon, for example, this not very good approximation.

Copernicus showed that by adding a third circle one could get a better approximation to the observed data. This suggests that if $\varphi(t)$ is the true motion of the moon, then there exist $r_1, r_2, \dots, r_n$ and $\lambda_1, \lambda_2, \dots, \lambda_n$ such that for all $t \in \mathbb{R}$,

$$\left|\varphi(t) - \sum_{j=1}^{n} r_j e^{i\lambda_j t}\right| \leq \varepsilon,$$

where $\varepsilon > 0$ is the observational error. If the numbers $\lambda_1, \lambda_2, \dots, \lambda_n$ are not all rational multiplies of one real number, then the finite sum is not periodic. It would be almost periodic in the sense of the following Condition: If for every positive $\varepsilon > 0$, there exists a finite sum

$$\sum_{j=1}^{n} r_j e^{i\lambda_j t} \equiv p(t)$$

and for all $t \in \mathbb{R}$,

$$|f(t) - p(t)| < \varepsilon,$$

then the function $f(t)$ is *almost periodic.*

The idea of Ptolemy and Copernicus was to show that the motion of a planet is described by functions of this type. The main aspects of the historical development of this problem can be found in [122, 179].

The formal theory of almost periodic functions was developed by Harald Bohr [26]. In this paper Bohr was interested in series of the form

$$\sum_{n=1}^{\infty} e^{-\lambda_n s}$$

called Dirichlet series, one of which is the Riemann-zeta function. In his researches he noticed that along the lines $Re(s) = const.$, these functions had a regular behavior. He apparently hoped that a formal study of this behavior might give him some insight into the distribution of values of the Dirichlet series. The regular behavior he discovered we shall consider in the following way [55].

The continuous function f is *regular*, if for every $\varepsilon > 0$ and for every $t \in \mathbb{R}$, the set $T(f, \varepsilon)$,

$$T(f,\varepsilon) \equiv \left\{\tau;\ \sup_{t\in\mathbb{R}} |f(t+\tau) - f(t)| < \varepsilon\right\}$$

is relatively dense in $\mathbb{R}$, i.e. if there is an $l > 0$ such that every interval of length l has a non-void intersection with $T(f, \varepsilon)$.

It is easy to see that the sum of two regular functions is also regular, and a uniform limit of functions from this class will converge to a regular function. Consequently, all regular functions are *almost periodic in the sense of Bohr.*

Later, Bohr considered the problem of when the integral of an almost periodic function is almost periodic. Many applications of this theory to various fields became known during the late 1920s. One of the results connected with the work of Bohr and Neugebauer [27] is that the bounded solutions of the system of differential equations in the form

$$\dot{x} = Ax + f$$

are almost periodic by necessity, when A is a scalar matrix, and f is almost periodic in the sense of Bohr.

The single most useful property of almost periodicity for studying differential equations is investigated from Bochner [22]. In this paper he introduced the following definition.

The continuous function f is *normal*, if from every sequence of real numbers $\{\alpha_n\}$ one can extract a subsequence $\{\alpha_{n_k}\}$ such that

$$\lim_{k\to\infty} f(t + \alpha_{n_k}) = g(t)$$

exists uniformly on $\mathbb{R}$. Bochner also proved the equivalence between the classes of *normal* and *regular* functions.

The utility of this definition for different classes of differential equations was exploited by Bochner in [23, 24].

Later on, in 1933 in Markoff's paper [114] on the study of almost periodic solutions of differential equations, it was recognized that almost periodicity and stability were closely related. Here for the fist time it was considered that strong stable bounded solutions are almost periodic.

After the first remarkable results in the area of almost periodicity in the middle of the twentieth century a number of impressive results were achieved. Some examples may be found are in the papers [12, 21, 47, 49, 53, 55–57, 69, 84, 97, 98, 143, 180, 183, 193–195, 201].

The beginning of the study of almost periodic piecewise continuous functions came in the 1960s. Developing the theory of impulsive differential equations further requires an introduction of definitions for these new objects. The properties of the classical (continuous) almost periodic functions can be greatly changed by impulses.

The first definitions and results in this new area were published by Halanay and Wexler [63], Perestyuk, Ahmetov and Samoilenko [8, 9, 127, 136–141], Hekimova and Bainov [67], Bainov, Myshkis, Dishliev and Stamov [17, 18].