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Regularity Estimates for Nonlinear Elliptic and Parabolic Problems

Cetraro, Italy 2009

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Preface

This volume collects the notes of the CIME course **Regularity Estimates for Nonlinear Elliptic and Parabolic Problems** held in Cetraro (Italy) on June 22–27, 2009. The school consisted in five series of lectures, delivered by

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The issue of regularity has obviously played a central role in the theory of Partial Differential Equations, almost since its inception, and despite the tremendous development, it still remains a very fruitful research field.

In particular regularity estimates for degenerate and singular elliptic and parabolic equations have developed considerably in the last years, in many unexpected and challenging directions.

Because of all these recent results, it seemed timely to trace an overview that would highlight emerging trends and issues of this fascinating research topic in a proper and effective way.

The course aimed at showing the deep connections among all these topics and at opening new research directions, through the contribution of leading experts in all these fields.

Emmanuele DiBenedetto gave a course on

Introduction to Regularity Theory for Degenerate Parabolic Equations in Divergence Form

discussing some techniques recently introduced to investigate the local and global behavior of solutions to degenerate parabolic equations when their principal part fails to be coercive. The equations have to be regarded in their own intrinsic geometry, and the solutions have a limited degree of regularity. DiBenedetto showed how identifying regularity classes as functions

of the degenerate and/or singular structure of the equation is part of an emerging theory which promises to yield an understanding of degeneracy and/or singularity in Partial Differential Equations. Unfortunately there are no notes of this course.

The course of *John Lewis* on

Applications of Boundary Harnack Inequalities
for p-Harmonic Functions and Related Topics

discussed applications of recent work and techniques concerning the boundary behavior of positive p-harmonic functions vanishing on a portion of the boundary of Lipschitz, chord arc, and Reifenberg flat domains. At first fundamental properties of p-harmonic functions and elliptic measure were presented. Then the dimension of p-harmonic measure was dealt with. The final part of the course first considered boundary Harnack inequalities and the Martin boundary problem in Reifenberg flat and Lipschitz domains, and at the end uniqueness and regularity both in free boundary and inverse type problems.

Peter Lindqvist presented in his course

Regularity of Supersolutions

a general theory for supersolutions of the p-Laplace Equation. Indeed the regularity theory for *solutions* to the parabolic p-laplacian is a well-developed topic, but when it comes to (semicontinuous) *supersolutions* and *subsolutions* a lot remains to be done. Supersolutions are often auxiliary tools as in the celebrated Perron method, for example, but they are also interesting in their own right. Therefore, the lectures were entirely focused on this important issue.

Juan J. Manfredi delivered a series of lectures on

Introduction to random Tug-of-War games and PDEs

providing an introduction to the connection between the theory of stochastic tug-of-war games and nonlinear equations of p-Laplacian type in the Euclidean and discrete cases. The fundamental contributions of Kolmogorov, Ito, Kakutani, Doob, Hunt, Lévy, and many others have shown the profound and powerful connection between classical linear potential theory and probability theory. The idea behind the classical interplay is that harmonic functions and martingales share a common cancellation property that can be expressed by using mean value properties. In his lectures, Manfredi showed how this approach turns out to be very useful in the nonlinear theory as well.

Sandro Salsa taught a course on

The Problems of the Obstacle in Lower Dimension
and for the Fractional Laplacian

giving a somewhat self-contained presentation of the results concerning the analysis of the solution and the free boundary of the thin obstacle problem and more generally of the obstacle for the fractional Laplacian. He started from the thin obstacle problem, considering the case of *zero obstacle.* In

this case, the main ideas and tools were clearly seen and developed without too many technicalities and in a somewhat self-contained fashion. Later, he extended the results on the optimal regularity and the analysis of the *regular part* of the free boundary to the general case for $(-\Delta)^s$.

This series of lectures attracted approximately 50 participants, largely PhD students or post-docs, and also senior researchers; we are sure that this CIME course was rich of useful suggestions and ideas for inspiring new developments, and opening new research prospects in the near future.

We wish to thank all the lecturers for their active participation and their valuable contribution, and the CIME foundation, in particular the director Prof. Pietro Zecca and the secretary Prof. Elvira Mascolo, for their helpful support and for the organization of such a remarkable event in Cetraro.

Pavia, Italy *Ugo Gianazza*
Lexington, KY *John Lewis*

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