

Lecture Notes in Mathematics

2044

Editors:

J.-M. Morel, Cachan

B. Teissier, Paris

For further volumes:

<http://www.springer.com/series/304>

Kendall Atkinson • Weimin Han

Spherical Harmonics and Approximations on the Unit Sphere: An Introduction

Kendall Atkinson
University of Iowa
Department of Mathematics
and Department of Computer Science
Iowa City, IA 52242
USA

Weimin Han
University of Iowa
Department of Mathematics
Iowa City, IA 52242
USA

ISBN 978-3-642-25982-1 e-ISBN 978-3-642-25983-8
DOI 10.1007/978-3-642-25983-8
Springer Heidelberg Dordrecht London New York

Lecture Notes in Mathematics ISSN print edition: 0075-8434
ISSN electronic edition: 1617-9692

Library of Congress Control Number: 2012931330

Mathematics Subject Classification (2010): 41A30, 65N30, 65R20

© Springer-Verlag Berlin Heidelberg 2012

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilm or in any other way, and storage in data banks. Duplication of this publication or parts thereof is permitted only under the provisions of the German Copyright Law of September 9, 1965, in its current version, and permission for use must always be obtained from Springer. Violations are liable to prosecution under the German Copyright Law.

The use of general descriptive names, registered names, trademarks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Preface

Spherical harmonics have been studied extensively and applied to solving a wide range of problems in the sciences and engineering. Interest in approximations and numerical methods for problems over spheres has grown steadily. These notes provide an introduction to the theory of spherical harmonics in an arbitrary dimension as well as a summarizing account of classical and recent results on some aspects of approximation by spherical polynomials and numerical integration over the sphere. The notes are intended for graduate students in the mathematical sciences and researchers who are interested in solving problems involving partial differential and integral equations on the sphere, especially on the unit sphere \mathbb{S}^2 in \mathbb{R}^3 . We also discuss briefly some related work for approximation on the unit disk in \mathbb{R}^2 , with those results being generalizable to the unit ball in more dimensions. The subject of theoretical approximation of functions on \mathbb{S}^d , $d > 2$, using spherical polynomials has been an active area of research over the past several decades. We summarize some of the major results, giving some insight into them; however, these notes are not intended to be a complete development of the theory of approximation of functions on \mathbb{S}^d by spherical polynomials.

There are a number of other approaches to the approximation of functions on the sphere. These include spline functions on the sphere, wavelets, and meshless discretization methods using radial basis functions. For a general survey of approximation methods on the sphere, see Fasshauer and Schumaker [46]; and for a more complete development, see Freedman et al. [47]. For more recent books devoted to radial basis function methods, see Buhmann [24], Fasshauer [45], and Wendland [118]. For a recent survey of numerical integration over \mathbb{S}^2 , see Hesse et al. [63].

During the preparation of the book, we received helpful suggestions from numerous colleagues and friends. We particularly thank Feng Dai (University of Alberta), Mahadevan Ganesh (Colorado School of Mines), Olaf Hansen

(California State University, San Marcos), and Yuan Xu (University of Oregon). We also thank the anonymous reviewers for their comments that have helped improve the final manuscript. This work was partially supported by a grant from the Simons Foundation (# 207052 to Weimin Han).

Contents

1	Preliminaries	1
1.1	Notations	3
1.2	The Γ -Function	5
1.3	Basic Results Related to the Sphere	6
2	Spherical Harmonics	11
2.1	Spherical Harmonics Through Primitive Spaces	11
2.1.1	Spaces of Homogeneous Polynomials	13
2.1.2	Legendre Harmonic and Legendre Polynomial	17
2.1.3	Spherical Harmonics	19
2.2	Addition Theorem and Its Consequences	20
2.3	A Projection Operator	26
2.4	Relations Among Polynomial Spaces	30
2.5	The Funk–Hecke Formula	34
2.6	Legendre Polynomials: Representation Formulas	36
2.6.1	Rodrigues Representation Formula	36
2.6.2	Integral Representation Formulas	39
2.7	Legendre Polynomials: Properties	41
2.7.1	Integrals, Orthogonality	42
2.7.2	Differential Equation and Distribution of Roots	43
2.7.3	Recursion Formulas	45
2.7.4	Generating Function	52
2.7.5	Values and Bounds	56
2.8	Completeness	60
2.8.1	Completeness in $C(\mathbb{S}^{d-1})$	60
2.8.2	Completeness in $C(\mathbb{S}^{d-1})$ via the Poisson Identity	64
2.8.3	Convergence of Fourier–Laplace Series	66
2.8.4	Completeness in $L^2(\mathbb{S}^{d-1})$	69
2.9	The Gegenbauer Polynomials	71

2.10	The Associated Legendre Functions	74
2.10.1	Definition and Representation Formulas	74
2.10.2	Properties	77
2.10.3	Normalized Associated Legendre Functions	80
2.11	Generating Orthonormalized Bases for Spherical Harmonic Spaces	81
3	Differentiation and Integration over the Sphere	87
3.1	The Laplace–Beltrami Operator	87
3.2	A Formula for the Laplace–Beltrami Operator	94
3.3	Spherical Harmonics As Eigenfunctions of the Laplace–Beltrami Operator	96
3.4	Some Integration Formulas	100
3.5	Some Differentiation Formulas	104
3.6	Some Integral Identities for Spherical Harmonics	106
3.7	Integral Identities Through the Funk–Hecke Formula	113
3.7.1	A Family of Integral Identities for Spherical Harmonics	114
3.7.2	Some Extensions	118
3.8	Sobolev Spaces on the Unit Sphere	119
3.9	Positive Definite Functions	124
4	Approximation Theory	131
4.1	Spherical Polynomials	132
4.2	Best Approximation on the Unit Sphere	135
4.2.1	The Approach to Best Approximation of Dai and Xu	136
4.2.2	The Approach to Best Approximation of Ragozin ...	142
4.2.3	Best Simultaneous Approximation Including Derivatives	149
4.2.4	Lebesgue Constants	150
4.2.5	Best Approximation for a Parameterized Family	152
4.3	Approximation on the Unit Disk	154
4.3.1	Orthogonal Polynomials	155
4.3.2	Properties of Orthogonal Polynomials over \mathbb{B}^2	161
4.3.3	Orthogonal Expansions	162
5	Numerical Quadrature	165
5.1	The Use of Univariate Formulas	166
5.2	Composite Methods	172
5.2.1	The Centroid Method	174
5.2.2	General Composite Methods	176
5.2.3	Error Analysis	178
5.3	High Order Gauss-Type Methods	185
5.3.1	Efficiency of a High-Order Formula	188
5.3.2	The Centroid Method	189
5.3.3	An Alternative Approach	190

5.4	Integration of Scattered Data	193
5.5	Integration of Singular Functions	195
5.5.1	Singular Integrands	199
5.6	Quadrature over the Unit Disk.....	201
5.6.1	A Product Gauss Formula.....	202
5.7	Discrete Orthogonal Expansions	203
5.7.1	Hyperinterpolation over \mathbb{S}^2	204
5.7.2	Hyperinterpolation over the Unit Disk	210
6	Applications: Spectral Methods	211
6.1	A Boundary Integral Equation	212
6.1.1	Convergence Theory	214
6.1.2	Quadrature	217
6.1.3	A Numerical Example	220
6.2	A Spectral Method for a Partial Differential Equation.....	224
6.2.1	Implementation	229
6.2.2	A Numerical Example	230
6.3	A Galerkin Method for a Beltrami-Type Equation.....	232
	References	237
	Index	243