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William J. Layton • Leo G. Rebholz

# Approximate Deconvolution Models of Turbulence

Analysis, Phenomenology  
and Numerical Analysis

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