

Editors:

J.-M. Morel, Cachan

B. Teissier, Paris

For further volumes:

<http://www.springer.com/series/304>



**FONDAZIONE
CIME
ROBERTO CONTI**

CENTRO INTERNAZIONALE MATEMATICO ESTIVO
INTERNATIONAL MATHEMATICAL SUMMER CENTER

Fondazione C.I.M.E., Firenze

C.I.M.E. stands for *Centro Internazionale Matematico Estivo*, that is, International Mathematical Summer Centre. Conceived in the early fifties, it was born in 1954 in Florence, Italy, and welcomed by the world mathematical community: it continues successfully, year for year, to this day.

Many mathematicians from all over the world have been involved in a way or another in C.I.M.E.'s activities over the years. The main purpose and mode of functioning of the Centre may be summarised as follows: every year, during the summer, sessions on different themes from pure and applied mathematics are offered by application to mathematicians from all countries. A Session is generally based on three or four main courses given by specialists of international renown, plus a certain number of seminars, and is held in an attractive rural location in Italy.

The aim of a C.I.M.E. session is to bring to the attention of younger researchers the origins, development, and perspectives of some very active branch of mathematical research. The topics of the courses are generally of international resonance. The full immersion atmosphere of the courses and the daily exchange among participants are thus an initiation to international collaboration in mathematical research.

C.I.M.E. Director

Pietro ZECCA

Dipartimento di Energetica "S. Stecco"

Università di Firenze

Via S. Marta, 3

50139 Florence

Italy

e-mail: zecca@unifi.it

C.I.M.E. Secretary

Elvira MASCOLO

Dipartimento di Matematica "U. Dini"

Università di Firenze

viale G.B. Morgagni 67/A

50134 Florence

Italy

e-mail: mascolo@math.unifi.it

For more information see CIME's homepage: <http://www.cime.unifi.it>

CIME activity is carried out with the collaboration and financial support of:

- INdAM (Istituto Nazionale di Alta Matematica)
- MIUR (Ministero dell'Università e della Ricerca)

Silvia Bertoluzza • Ricardo H. Nochetto
Alfio Quarteroni • Kunibert G. Siebert
Andreas Veeger

Multiscale and Adaptivity: Modeling, Numerics and Applications

C.I.M.E. Summer School,
Cetraro, Italy 2009

Editors:
Giovanni Naldi
Giovanni Russo

 Springer



Silvia Bertoluzza
CNR
Istituto di Matematica Applicata
e Tecnologie Informatiche
Pavia
Italy

Ricardo H. Nochetto
University of Maryland
Department of Mathematics
College Park, MD
USA

Alfio Quarteroni
École Polytechnique Fédérale
de Lausanne
Chaire de Modelisation
et Calcul Scientifique (CMCS)
Lausanne
Switzerland

Kunibert G. Siebert
Universität Stuttgart
Fakultät für Mathematik und Physik
Stuttgart
Germany

Andreas Veese
Università degli Studi di Milano
Dipartimento di Matematica
Milano
Italy

ISBN 978-3-642-24078-2 e-ISBN 978-3-642-24079-9
DOI 10.1007/978-3-642-24079-9
Springer Heidelberg Dordrecht London New York

Lecture Notes in Mathematics ISSN print edition: 0075-8434
ISSN electronic edition: 1617-9692

Library of Congress Control Number: 2011943495

Mathematics Subject Classification (2010): 65M50, 65N50, 65M55, 65T60, 65N30, 65M60, 76MXX

© Springer-Verlag Berlin Heidelberg 2012

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilm or in any other way, and storage in data banks. Duplication of this publication or parts thereof is permitted only under the provisions of the German Copyright Law of September 9, 1965, in its current version, and permission for use must always be obtained from Springer. Violations are liable to prosecution under the German Copyright Law.

The use of general descriptive names, registered names, trademarks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Preface

The CIME-EMS Summer School in applied mathematics on “Multiscale and Adaptivity: Modeling, Numerics and Applications” was held in Cetraro (Italy) from July 6 to 11, 2009. This course has focused on mathematical methods for systems that involve multiple length/time scales and multiple physics. The complexity of the structure of these systems requires suitable mathematical and computational tools. In addition, mathematics provides an effective approach toward devising computational strategies for handling multiple scales and multiple physics. This course brought together researchers and students from different areas such as partial differential equations (PDEs), analysis, mathematical physics, numerical analysis, and scientific computing to address the challenges present in these issues. Physical, chemical, and biological processes for many problems in computational physics, biology, and material science span length and time scales of many orders of magnitude. Traditionally, scientists and research groups have focused on methods that are particularly applicable in only one regime, and knowledge of the system at one scale has been transferred to another scale only indirectly. Microscopic models, for example, have been often used to find the effective parameters of macroscopic models, but for obvious computational reasons, microscopic and macroscopic scales have been treated separately.

The enormous increase in computational power available (due to the improvement both in computer speed and in efficiency of the numerical methods) allows in some cases the treatment of systems involving scales of different orders of magnitude, arising, for example, when effective parameters in a macroscopic model depend on a microscopic model, or when the presence of a singularity in the solution produces a continuum of length scales. However, the numerical solution of such problems by classical methods often leads to an inefficient use of the computational resources, even up to the point that the problem cannot be solved by direct numerical simulation. The main reasons for this are that the necessary resolution of a fine scale entails an over-resolution of coarser scales, the position of the singularity is not known beforehand, the gap between the scales is too big for a treatment in the same framework. In other cases, the structure of the mathematical models that treat the system at the different scales varies a lot, and therefore new mathematical techniques

are required to treat systems described by different mathematical models. Finally, in many cases one is interested in the accurate treatment of a small portion of a large system, and it is too expensive to treat the whole system at the required accuracy. In such cases, the region of interest is modeled and discretized with great accuracy, while the remaining parts of the system are described by some reduced model, which enormously simplifies the calculation, still providing reasonable boundary conditions for the region of interest, allowing the required level of detail in such region.

The outstanding and internationally renowned lecturers have themselves contributed in an essential way to the development of the theory and techniques that constituted the subjects of the courses. The selection of the five topics of the CIME-EMS Course was not an easy task because of the wide spectrum of recent developments in multiscale methods and models. The six world leading experts illustrated several aspects of the multiscale approach.

Silvia Bertoluzza, from IMATI-CNR Pavia, described the concept of nonlinear sparse wavelet approximation of a given (known) function. Next she showed how the tools just introduced can be applied in order to write down efficient adaptive schemes for the solution of PDEs.

Bjorn Engquist, from ICES University of Texas at Austin, gradually guided the audience toward the realm of “Multiscale Modeling,” by providing mathematical ground for state-of-the-art analytical and numerical multiscale problems.

Alfio Quarteroni, from EPFL, Lausanne, and Politecnico di Milano, considered adaptivity in mathematical modeling for the description and simulation of complex physical phenomena. He showed that the combination of hierarchical mathematical models can be set up with the aim of reducing the computational complexity in the real life problems.

Ricardo H. Nochetto, from University of Maryland, and Andreas Veiser, from Università di Milano, in their joint course started with an overview of the a posteriori error estimation for finite element methods, and then they exposed recent results about the convergence and complexity of adaptive finite element methods.

Kunibert G. Siebert, from Universität Duisburg-Essen, described the implementation of adaptive finite element methods using toolbox ALBERTA (created by Alfred Schmidt and Kunibert G. Siebert, which is freely available).

The main “senior” lecturers were complemented by four young speakers, who gave account of detailed examples or applications during an afternoon session dedicated to them. Matteo Semplice, Università dell’Insubria, has spoken about “Numerical entropy production and adaptive schemes for conservation laws,” Tiziano Passerini, from Emory University, about “A 3D/1D geometrical multiscale model of cerebral vasculature,” Loredana Gaudio, MOX Politecnico di Milano, about “Spectral element discretization of optimal control problems,” and Carina Geldhauser, Universität Tuebingen, described “A discrete-in-space scheme converging to an unperturbed Cahn–Hilliard equation.” Both the lectures and the active interactions with and within the audience contributed to the scientific success of the course, which was attended by about 60 people of various nationality (14 different countries), ranging from first year PhD students to full professors. The present

volume collects the expanded version of the lecture notes by Silvia Bertoluzza, Alfio Quarteroni (with Marco Discacciati and Paola Gervasio as coauthors), Ricardo H. Nochetto, Andreas Veese, and Kunibert G. Siebert. We are grateful to them for such high quality scientific material.

As editors of these Lecture Notes and as scientific directors of the course, we would like to thank the many persons and Institutions that contributed to the success of the school. It is our pleasure to thank the members of the Scientific Committee of CIME for their invitation to organize the School; the Director, Prof. Pietro Zecca, and the Secretary, Prof. Elvira Mascolo, for their efficient support during the organization and their generous help during the school. We were particularly pleased by the fact that the European Mathematical Society (EMS) chose to cosponsor this CIME course as one of its Summer School in applied mathematics for 2009. Our special thanks go to the lecturers for their early preparation of the material to be distributed to the participants, for their excellent performance in teaching the courses and their stimulating scientific contributions. All the participants contributed to the creation of an exceptionally friendly atmosphere in the beautiful environment around the School. We also wish to thank Dipartimento di Matematica of the Università degli Studi di Milano, and Dipartimento di Matematica ed Informatica of the Università degli Studi di Catania for their financial support.

Catania
Milano

Giovanni Naldi
Giovanni Russo

Contents

Adaptive Wavelet Methods	1
Silvia Bertoluzza	
1 Introduction	1
2 Multiresolution Approximation and Wavelets	2
2.1 Riesz Bases	2
2.2 Multiresolution Analysis	3
2.3 Examples	9
2.4 Beyond $L^2(\mathbb{R})$	17
3 The Fundamental Property of Wavelets	21
3.1 The Case $\Omega = \mathbb{R}$: The Frequency Domain Point of View vs. the Space Domain Point of View	22
4 Adaptive Wavelet Methods for PDE's: The First Generation	34
4.1 The Adaptive Wavelet Collocation Method	37
5 The New Generation of Adaptive Wavelet Methods	40
5.1 A Posteriori Error Estimates	41
5.2 Nonlinear Wavelet Methods for the Solution of PDE's	46
5.3 The CDD2 Algorithm	48
5.4 Operations on Infinite Matrices and Vectors	51
References	54
Heterogeneous Mathematical Models in Fluid Dynamics and Associated Solution Algorithms	57
Marco Discacciati, Paola Gervasio, and Alfio Quarteroni	
1 Introduction and Motivation	57
2 Variational Formulation Approach	67
2.1 The Advection–Diffusion Problem	67
2.2 Variational Analysis for the Advection–Diffusion Equation	68
2.3 Domain Decomposition Algorithms for the Solution of the Reduced Advection–Diffusion Problem	72
2.4 Numerical Results for the Advection–Diffusion Problem	77

2.5	Navier–Stokes/Potential Coupled Problem	80
2.6	Asymptotic Analysis of the Coupled Navier–Stokes/ Darcy Problem	82
2.7	Solution Techniques for the Navier–Stokes/Darcy Coupling	85
2.8	Numerical Results for the Navier–Stokes/Darcy Problem	90
3	Virtual Control Approach	94
3.1	Virtual Control Approach Without Overlap for AD Problems	95
3.2	Domain Decomposition with Overlap	105
3.3	Virtual Control Approach with Overlap for the Advection–Diffusion Equation	108
3.4	Virtual Control with Overlap for the Stokes–Darcy Coupling	114
3.5	Coupling for Incompressible Flows	119
	References	120
	Primer of Adaptive Finite Element Methods	125
	Ricardo H. Nochetto and Andreas Veiser	
1	Piecewise Polynomial Approximation	125
1.1	Classical vs Adaptive Pointwise Approximation	126
1.2	The Sobolev Number: Scaling and Embedding	127
1.3	Conforming Meshes: The Bisection Method	129
1.4	Finite Element Spaces	133
1.5	Polynomial Interpolation in Sobolev Spaces	134
1.6	Adaptive Approximation	139
1.7	Nonconforming Meshes	143
1.8	Notes	145
1.9	Problems	146
2	Error Bounds for Finite Element Solutions	148
2.1	Model Boundary Value Problem	148
2.2	Galerkin Solutions	149
2.3	Finite Element Solutions and A Priori Bound	150
2.4	A Posteriori Upper Bound	151
2.5	Notes	157
2.6	Problems	158
3	Lower A Posteriori Bounds	159
3.1	Local Lower Bounds	160
3.2	Global Lower Bound	166
3.3	Notes	167
3.4	Problems	168
4	Convergence of AFEM	170
4.1	A Model Adaptive Algorithm	171
4.2	Convergence	172
4.3	Notes	178
4.4	Problems	179

5	Contraction Property of AFEM	180
5.1	Modules of AFEM for the Model Problem	180
5.2	Basic Properties of AFEM	182
5.3	Contraction Property of AFEM	185
5.4	Example: Discontinuous Coefficients	189
5.5	Extensions and Restrictions	191
5.6	Notes	193
5.7	Problems	193
6	Complexity of Refinement	194
6.1	Chains and Labeling for $d = 2$	195
6.2	Recursive Bisection	197
6.3	Conforming Meshes: Proof of Theorem 1	199
6.4	Nonconforming Meshes: Proof of Lemma 3	204
6.5	Notes	205
6.6	Problems	206
7	Convergence Rates	206
7.1	The Total Error	207
7.2	Approximation Classes	208
7.3	Quasi-Optimal Cardinality: Vanishing Oscillation	212
7.4	Quasi-Optimal Cardinality: General Data	215
7.5	Extensions and Restrictions	218
7.6	Notes	221
7.7	Problems	221
	References	223

Mathematically Founded Design of Adaptive Finite Element

Software	227
Kunibert G. Siebert	
1 Introduction	227
1.1 The Variational Problem	229
1.2 The Basic Adaptive Algorithm	230
2 Triangulations and Finite Element Spaces	232
2.1 Triangulations	232
2.2 Finite Element Spaces	234
2.3 Basis Functions and Evaluation of Finite Element Functions	240
2.4 ALBERTA Realization of Finite Element Spaces	244
3 Refinement By Bisection	246
3.1 Basic Thoughts About Local Refinement	246
3.2 Bisection Rule: Bisection of One Single Simplex	248
3.3 Triangulations and Refinements	252
3.4 Refinement Algorithms	255
3.5 Complexity of Refinement By Bisection	260
3.6 ALBERTA Refinement	262
3.7 Mesh Traversal Routines	263

4	Assemblage of the Linear System	268
4.1	The Variational Problem and the Linear System	269
4.2	Assemblage: The Outer Loop	272
4.3	Assemblage: Element Integrals	276
4.4	Remarks on Iterative Solvers	283
5	The Adaptive Algorithm and Concluding Remarks	285
5.1	The Adaptive Algorithm	286
5.2	Concluding Remarks	294
6	Supplement: A Nonlinear and a Saddlepoint Problem	297
6.1	The Prescribed Mean Curvature Problem in Graph Formulation	297
6.2	The Generalized Stokes Problem	301
	References	308
	List of Participants	311