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Olaf Post

Spectral Analysis on Graph-Like Spaces

Olaf Post
Durham University
Department of Mathematical Sciences
Science Laboratories
South Road
Durham DH1 3LE
United Kingdom
olaf.post@durham.ac.uk

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Preface

In this monograph, we analyse thin tubular structures, so-called “graph-like spaces”, and their natural limits, when the radius of a graph-like space tends to zero. The limit space is typically a metric graph, i.e. a graph, where each edge is associated a length, and therefore, the space turns into a one-dimensional manifold with singularities at the vertices. On both, the graph-like spaces and the metric graph, we can naturally define Laplace-like differential operators. We are interested in asymptotic properties of such operators. In particular, we show norm resolvent convergence, convergence of the spectra and resonances.

Tubular structures with small radius have attracted a lot of attention in the last years. Tubular structures are frequently used in different areas such as mathematical physics to describe properties of nano-structures, in spectral geometry to provide examples with given spectral properties, or in global analysis to calculate spectral invariants.

Since the underlying spaces in the thin radius limit change, and even become singular in the limit, we develop new tools such as

- Norm convergence of operators acting in different Hilbert spaces.
- An extension of the concept of boundary triples to partial differential operators.
- An abstract definition of resonances via boundary triples.

These tools are formulated in an abstract framework, independent of the original problem of graph-like spaces, and in a way that they may be applied in many other situations, for example when the underlying space is geometrically perturbed.

We briefly outline the content of this work. Chapter 1 is devoted to an exemplary overview of the results proven in this work, their history and a discussion of further research. In Chap. 2, we introduce the necessary concepts for discrete and metric graphs, especially Laplace-type operators. Chapter 3 contains partially new material, in particular, boundary triples associated with a quadratic form with applications to the PDE case. Moreover, we derive an abstract version of complex scaling via boundary triples and introduce resonances, i.e. poles of a meromorphic continuation of the resolvent. Chapter 4 provides new material on (norm-)convergence of operators and quadratic forms in different Hilbert spaces. We

extend these concepts to non-self-adjoint operators. Chapter 5 provides perturbation arguments for manifolds under a change of the Riemannian metric. Moreover, we present how a tubular neighbourhood can be reduced to the underlying one-dimensional space using separation of variables.

In Chap. 6, we define abstract graph-like manifolds associated with a star-shaped graph. We allow different scaling behaviours of the vertex neighbourhood and different boundary conditions, leading to different limit operators. Finally, in Chap. 7, we combine the convergence results for star-shaped graphs in order to get convergence results for general metric graphs and graph-like manifolds. We use the language of boundary triples in order to combine the local convergence results for star-graphs to convergence results for general graphs. Combining and extending existing results in the literature, we show norm-resolvent convergence of the corresponding Laplacians, convergence of the spectrum and of resonances.

Berlin

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Contents

1	Introduction	1
1.1	About This Monograph	1
1.1.1	Convergence of Laplacians on Graph-Like Spaces	1
1.1.2	Tools from Functional Analysis and Operator Theory	2
1.1.3	Outline of the Work	3
1.1.4	Related Topics Not Included in This Work	4
1.2	History, Results and Motivation	5
1.2.1	Convergence of Laplacians on Graph-Like Spaces: The Neumann Case	6
1.2.2	Convergence of Laplacians on Graph-Like Spaces: The Dirichlet Case	7
1.2.3	Convergence of Resonances	9
1.2.4	Mathematical Physics	11
1.2.5	Models from Mathematical Biology	15
1.2.6	Spectral Geometry and Spectral Invariants	15
1.2.7	Global Analysis	17
1.2.8	Convergence of Operators Acting in Different Spaces	19
1.2.9	Boundary Triples	22
1.3	Convergence of Operators and Spectra: A Brief Overview	25
1.3.1	Graph-Like Spaces	26
1.3.2	The Limit Hilbert Spaces Associated to the Graph Models	29
1.3.3	Convergence Results for Operators in Different Hilbert Spaces	32
1.3.4	Convergence Results for Graph-Like Spaces	33
1.4	Boundary Triples and Convergence of Resonances: A Brief Overview	35
1.4.1	Boundary Triples Associated with Quadratic Forms	35
1.4.2	Resonances and Complex Dilation	40
1.4.3	Convergence of Resonances on Graph-Like Spaces	44

1.5	Consequences of the Convergence Results	46
1.5.1	Spectral Gaps and Covering Manifolds	46
1.5.2	Eigenvalues in Gaps	48
1.5.3	Equilateral Graphs.....	49
1.5.4	Spectral Band Edges	52
1.5.5	The Decoupled Case	52
1.6	Outlook and Remarks	53
1.6.1	Geometric Perturbations	53
1.6.2	Scattering Properties	54
1.6.3	Convergence of Differential Forms and First Order Operators	54
1.6.4	Convergence of Boundary Triples Coupled via Graphs ...	55
1.6.5	Metric Structure on the Space of Operators.....	55
1.6.6	Dirichlet-to-Neumann Map and Inverse Problems.....	56
1.6.7	Fractal Metric Graphs	56
2	Graphs and Associated Laplacians	57
2.1	Discrete Graphs and Generalised Laplacians	58
2.1.1	Discrete Graphs and Vertex Spaces.....	58
2.1.2	Operators Associated with Vertex Spaces	64
2.2	Metric Graphs, Quantum Graphs and Associated Operators.....	68
2.2.1	Metric Graphs	69
2.2.2	Operators on Metric Graphs	70
2.2.3	Boundary Triples Associated with Quantum Graphs	76
2.3	Extended Quantum Graphs	81
2.4	Spectral Relations Between Discrete and Metric Graphs	84
2.4.1	Spectral Relation for Equilateral Graphs	84
2.4.2	Spectral Relation at the Bottom of the Spectrum	87
2.5	Some Trace Formulas on Metric and Discrete Graphs	91
3	Scales of Hilbert Space and Boundary Triples	97
3.1	Sesquilinear Forms, Associated Operators and Dense Subspaces	97
3.2	Scale of Hilbert Spaces Associated with a Non-negative Operator	102
3.3	Scale of Hilbert Spaces Associated with a Closed Operator	104
3.3.1	Scale of Hilbert Spaces of Second Order Associated with a Closed Operator	104
3.3.2	Scale of Hilbert Spaces of First Order Associated with a Closed Operator	107
3.4	Boundary Triples and Abstract Elliptic Theory	114
3.4.1	Boundary Triples Associated with Quadratic Forms	116
3.4.2	Elliptic Boundary Triples	125
3.4.3	Relation with Other Concepts of Boundary Triples and Examples	132
3.4.4	Krein's Resolvent Formulas and Spectral Relations	136

3.5	Half-Line Boundary Triples and Complex Dilation	140
3.5.1	Half-Line Boundary Triples	141
3.5.2	Complex Dilation.....	146
3.6	Coupled Boundary Triples and Dilation	152
3.6.1	Coupled Boundary Triples	153
3.6.2	Dilated Coupled Boundary Triples	158
3.7	Complexly Dilated Coupled Operators	164
3.7.1	Holomorphic Dependency	165
3.7.2	The Complexly Dilated Coupled Operator	167
3.7.3	The Complexly Dilated Coupled Operator on the First Order Spaces	170
3.8	Resonances	174
3.9	Boundary Maps and Triples Coupled via Graphs	180
4	Two Operators in Different Hilbert Spaces	187
4.1	Quasi-Unitary Identification Operators	188
4.2	Convergence of Self-Adjoint Operators in Different Hilbert Spaces	194
4.3	Spectral Convergence for Non-negative Operators	204
4.4	Convergence of Quadratic Forms in Different Hilbert Spaces	209
4.5	Convergence of Non-self-adjoint Operators	221
4.6	Spectral Convergence for Non-self-adjoint Operators	230
4.7	Convergence of Non-symmetric Forms	233
4.8	Closeness of Coupled Boundary Maps.....	242
4.9	Convergence of Resonances	246
5	Manifolds, Tubular Neighbourhoods and Their Perturbations	259
5.1	Manifolds	260
5.1.1	Manifolds with Boundary	260
5.1.2	Manifolds Constructed from Building Blocks	261
5.1.3	Laplacians and Quadratic Forms	262
5.1.4	Basic Estimates	264
5.1.5	Some Scaling Behaviour	266
5.2	Perturbations of the Metric	268
5.3	Tubular Neighbourhoods	273
5.3.1	Perturbations of the Product Structure	273
5.3.2	Shortened Edge Neighbourhoods	276
5.4	Embedded Tubular Neighbourhoods	278
5.5	Tubular Neighbourhoods with Neumann Boundary Conditions	280
5.6	Tubular Neighbourhoods with Dirichlet Boundary Conditions	284
6	Plumber's Shop: Estimates for Star Graphs and Related Spaces.....	291
6.1	The Graph Models for Neumann Boundary Conditions	292
6.1.1	Fast Decaying Vertex Volume	293
6.1.2	Slowly Decaying Vertex Volume	297
6.1.3	The Borderline Case.....	299

6.2	The Manifold Models	301
6.2.1	A Simple Graph-Like Manifold	301
6.2.2	Graph-Like Manifolds with Different Scalings	306
6.2.3	The Associated Quadratic Form, Operator and Boundary Triple	314
6.2.4	Manifolds with Infinite Ends	316
6.3	Some Vertex Neighbourhoods Estimates	316
6.4	Fast Decaying Vertex Neighbourhoods	324
6.5	Slowly Decaying Vertex Neighbourhoods	329
6.6	The Borderline Case: Reduction to the Graph Model	332
6.7	The Embedded Case	335
6.7.1	Embedded Graph-Like Spaces	335
6.7.2	Reduction to the Graph Model	338
6.8	Slowly Decaying and Borderline Case for Arbitrary Transversal Manifolds	342
6.8.1	The Enlarged Vertex Neighbourhood with Added Truncated Cones	342
6.8.2	Some More Vertex Neighbourhood Estimates	344
6.9	Slowly Decaying and Arbitrary Transversal Manifolds	348
6.10	The Borderline Case with Arbitrary Transversal Manifolds	352
6.11	Dirichlet Boundary Conditions: The Decoupling Case	354
6.11.1	The Graph and Manifold Models	354
6.11.2	Some Vertex and Edge Neighbourhoods Estimates	357
6.11.3	Reduction to the Graph Model	359
6.11.4	The Embedded Case	362
6.11.5	The Spectral Vertex Neighbourhood Condition	364
7	Global Convergence Results	367
7.1	Spectral Convergence for Graph-Like Spaces	368
7.1.1	Fast Decaying Vertex Volume	369
7.1.2	Slowly Decaying Vertex Volume	371
7.1.3	The Borderline Case	372
7.1.4	The Dirichlet Decoupled Case	374
7.1.5	The Embedded Case	375
7.2	Convergence of Resonances	377
7.2.1	Fast Decaying Vertex Volume	379
7.2.2	Slowly Decaying Vertex Volume	383
7.2.3	The Borderline Case	384
7.2.4	The Dirichlet Decoupled Case	385
A	Appendix	389
A.1	Convergence of Set Sequences	389
A.2	Estimates on Abstract Fibred Spaces	391
A.2.1	Vector-Valued Integrals	391

A.2.2	Fibred Spaces Over an Interval	393
A.2.3	Examples: Cones and Cylinders	398
References	407
Notation	419
Index	421