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Milnor Fiber Boundary of a Non-isolated Surface Singularity

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to

Miksa, Marcella, Balázs, Jankó

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In some of our computations we used the computer program *Mathematica*.

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