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Pseudo-periodic Maps and Degeneration of Riemann Surfaces

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Dedicated with respect and affection to the memory of Professor Itiro Tamura (1926–1991)

Preface

In 1944, Nielsen introduced a certain type of mapping classes of a surface which were called by him *surface transformation classes of algebraically finite type*, [53]. He introduced this type of mapping classes as a generalization of the mapping classes of finite order. By the celebrated Nielsen Theorem [52], the latter classes contain surface homeomorphisms of finite order (For a generalization, see Kerckhoff [30]). A mapping class of algebraically finite type does not necessarily contain a homeomorphism of finite order, but using Nielsen's theorem [52], one can show that it contains a homeomorphism f satisfying the following conditions (in what follows f will be an orientation-preserving homeomorphism of a closed, connected, oriented surface of genus g, Σ_g):

1. There exists a disjoint union of simple closed curves (which will be called *cut curves*)
$$\mathscr{C} = C_1 \cup C_2 \cup \cdots \cup C_r$$
on Σ_g such that $f(\mathscr{C}) = \mathscr{C}$, and
2. the restriction of f to the complement of $\mathscr{C}$,
$$f|(\Sigma_g - \mathscr{C}) : \Sigma_g - \mathscr{C} \to \Sigma_g - \mathscr{C}$$

is isotopic to a periodic map, namely a homeomorphism of finite order. (Cf. [53, Sect. 14], [22]).

In the present memoir, such a homeomorphism (and also a homeomorphism which is isotopic to such a homeomorphism) will be called a *pseudo-periodic map*. A periodic map is a special case of a pseudo-periodic map. In recent terminology, a homeomorphism f is pseudo-periodic if and only if either it is of finite order or its mapping class $[f]$ is reducible and all the component mapping classes are of finite order. (See [12,16,22,24,63]). A surface transformation class of algebraically finite type is nothing but a mapping class of a pseudo-periodic map.

Nielsen [53] studied these classes extensively and defined several important invariants, for instance, the *screw number* of f about a cut curve C_i which measures

the amount of the (fractional) Dehn-twist performed by a certain power f^α of f sending C_i to itself; or the character of C_i: whether it is "amphidrome" or not. Here C_i is *amphidrome* if there is an integer γ such that

$$f^\gamma(\overrightarrow{C_i}) = -\overrightarrow{C_i}.$$

He asserted in [53] that his invariants were a complete set of conjugacy invariants, meaning that if two pseudo-periodic maps

$$f_1 : \Sigma_g^{(1)} \to \Sigma_g^{(1)}$$

and

$$f_2 : \Sigma_g^{(2)} \to \Sigma_g^{(2)}$$

have these same invariants, then their mapping classes $[f_1]$ and $[f_2]$ are equivalent under a certain homeomorphism

$$h : \Sigma_g^{(1)} \to \Sigma_g^{(2)},$$

i.e. $[f_1] = [h^{-1} f_2 h]$. (For an exact formulation, see [22, Theorem 13.4]). However, his proof of this assertion was rather vague, and we need an invariant (the action of monodromy on the partition graph) which he did not mention explicitly. See Examples 6.3 and 6.4 in Chap. 6.

A pseudo-periodic map f is said to be *of negative twist* if the screw numbers about a certain system of cut curves are all negative (Chap. 3). The purpose of Part I of the present memoir is to construct a complete set of conjugacy invariants for a pseudo-periodic map f of negative twist.

We have added to Nielsen's invariants one more: *the action of f on the "partition graph"*, which is the action, induced by f, on the configuration of the partition of Σ_g obtained by cutting Σ_g along a certain system of cut curves $\{C_i\}_{i=1}^r$. The main result of Part I is roughly stated as follows (see Theorem 6.1 and 6.3 for precise statements):

Theorem 0.1. *Let $f_1 : \Sigma_g^{(1)} \to \Sigma_g^{(1)}$ and $f_2 : \Sigma_g^{(2)} \to \Sigma_g^{(2)}$ be pseudo-periodic maps of negative twist. Suppose that they have the same values in Nielsen's invariants and that their actions on the respective partition graphs are equivariantly isomorphic. Then there exists an orientation preserving homeomorphism $h : \Sigma_g^{(1)} \to \Sigma_g^{(2)}$ such that $[f_1] = [h^{-1} f_2 h]$.*

In the course of the proof, we develop (in Chaps. 3–5) the theory of *generalized quotients*, which are naturally associated with pseudo-periodic maps, just as ordinary quotient spaces are associated with periodic maps. This makes our proof of Theorem 0.1 unexpectedly long, but the generalized quotients will play an essential role also in the study of the degeneration of Riemann surfaces (in Part II). This was the main reason of our investigation, which therefore concentrated in the study

of generalized quotients. As a matter of fact, Theorem 0.1 above is just a (non immediate) corollary of our research.

The organization of Part I is as follows:

In Chap. 1, we review some basic results of Nielsen from [51, 53].

In Chap. 2, we define the "standard form" of a pseudo-periodic map f. Nielsen [53, Sect. 14] constructed a special homeomorphism which served as a standard form, but our standard form is slightly different from his. We prove the existence and the essential uniqueness of the homeomorphism in standard form which is isotopic to a given pseudo-periodic map (Theorem 2.1).

In Chap. 3, we introduce the notion of generalized quotients, and in particular, of *minimal quotients* which are the special case of generalized quotients that satisfy a certain "minimality condition". According to the definition given in Chap. 3, in order to have a generalized quotient, a pseudo-periodic map f must be in a very special form which we would like to call "superstandard form". It will be proved that any pseudo-periodic map f of negative twist is isotopic to a pseudo-periodic map in superstandard form having a minimal quotient (Theorem 3.1).

In Chap. 4, the following essential uniqueness will be proved (Theorem 4.1): suppose f_1 and $f_2 : \Sigma_g \to \Sigma_g$ are pseudo-periodic maps of negative twist, both in superstandard form. If they are homotopic, then their respective minimal quotients are isomorphic.

By the above existence and uniqueness theorems, we can generalize the definition of minimal quotients to cover any pseudo-periodic map of negative twist not necessarily in superstandard form, i.e., the minimal quotient of a pseudo-periodic map f of negative twist is constructed by first isotoping f to the superstandard form f' and then taking the minimal quotient of f', which is declared to be the minimal quotient of f.

The minimal quotient captures all of the Nielsen invariants constructed in [53]. Moreover, it will be proved in Part II that the "base space" of the minimal quotient of a pseudo-periodic map f of negative twist is homeomorphic to a (normally minimal) singular fiber of a one-parameter family of Riemann surfaces of genus g around which the topological monodromy is equivalent to $[f]$.

In Chap. 5, we prove a theorem in elementary number theory, which is basic to the arguments in Chaps. 3 and 4.

In Chap. 6, we consider the partition graph and the action of f on it. This action, together with the minimal quotient, determines the conjugacy class of $[f]$ in $\mathscr{M}_g$(Theorem 6.1). This result is further reformulated in terms of certain cohomology of "weighted graphs" (Theorem 6.3).

In Appendix A, we will give a proof of the following theorem: *let f and f' be (orientation- preserving) periodic maps of a compact surface Σ each component of which has negative Euler characteristic. Suppose f and $f' : (\Sigma, \partial\Sigma) \to (\Sigma, \partial\Sigma)$ are homotopic as maps of pairs. Then there exists a homeomorphism $h : \Sigma \to \Sigma$ isotopic to the identity, such that $f = h^{-1} f' h$.*

This theorem is used in the proof of Theorem 2.1. Among specialists, this theorem seems folklore. A. Edmonds informed, in a letter to the second named

author, that C. Frohman had proved a stronger result which implied the above theorem. Unfortunately, the authors could not find any reference giving an explicit proof, so we decided to write this appendix.

Pseudo-periodic maps of negative twist are closely related to the degeneration of Riemann surfaces. In fact, the topological monodromy around a singular fiber in a one-parameter family of Riemann surfaces is a pseudo-periodic map of negative twist (see [19], also [26, 58]).

In Part II of this memoir, we will apply the results in Part I to the topology of degeneration of Riemann surfaces. The main result of Part II is roughly summarized as follows:

Theorem 0.2. *The topological types of minimal degenerating families of Riemann surfaces of genus $g \geq 2$, over a disk, which are nonsingular outside the origin, are in a bijective correspondence with the conjugacy classes in the mapping class group $\mathscr{M}_g$ represented by pseudo-periodic maps of negative twist. The correspondence is given by the topological monodromy.*

In the case of $g = 1$, the validity of Theorem 0.2 is reduced by half: By Kodaira's classification [32] of singular fibers for genus 1, we see that every pseudo-periodic mapping class (of negative twist) of a torus can be realized as the topological monodromy of a singular fiber. Thus the correspondence is "surjective", but it is not "injective". For example, all the multiple fibers of type ${}_m\mathrm{I}_0$ (in Kodaira's notation) have the identity mapping class as their topological monodromy.

The assumption $g \geq 2$ is used almost everywhere in our proof: The existence of an admissible system of cut curves subordinate to a pseudo-periodic map (Lemma 1.1) is essential to the definition of various invariants of the map, and the proof of the existence requires $g \geq 2$. Also "homotopy implies conjugacy" theorem for periodic maps assumes $g \geq 2$, because in the proof we apply the hyperbolic geometry (see Appendix A). This theorem is indispensable in the proof of the uniqueness of the standard form (see Theorem 2.1 (ii)). Our arguments in later chapters depend on this uniqueness theorem.

We have tried to make the memoir as self-contained as possible, except for the two quotations from [51, 53]. (Theorems 1.1 and 1.2 of the present memoir). All the other arguments are elementary.

The authors are grateful to Allan Edmonds, Takayuki Oda and Hiroshige Shiga for their useful information and comments.

This work started during the first named author's first visit to Spain (1988) and was completed during his second visit (1991). The first named author would like to express his warmest thanks to the members and staffs of Facultad de Ciencias Matemáticas, Universidad Complutense de Madrid, for their kind hospitality.

Finally but not least at all, the authors deeply thank Srta. María Angeles Bringas for her benevolence, patience, and excellent skill shown in typing this memoir, without which it could have never been published.

ADDED:SEPTEMBER,2009[1]

The main body of the manuscript of the present memoir was completed in December 1991, and some remaining additional parts in January, 1992. Since then we have not found any occasion to publish this work, for several reasons; the length certainly was one. Another reason, but probably the main one, was the authors' inability to use Tex.

After a long delay of almost two decades, the authors find some unsatisfactory points in the manuscript, for example, it contains too many details, which might be a hindrance for readers who want to get a quick view, but on the other hand, it might help them to understand the details of the argument. Anyway the authors needed to compile these long (elementary and sometimes seemingly trivial) arguments to complete the proof of our theorems. Therefore, we have decided to keep the manuscript in its original form, except for the numbering of the chapters, theorems, propositions, figures, etc. A change we have made is the unification of the two different bibliographies, which were separately attached to each part, into one bibliography at the end. Also we added some references that were published after the completion of our manuscript and some more that we had missed involuntarily or were unknown to us due to our limitations. Unfortunately, the authors cannot be sure even now of the completeness of the augmented bibliography.

A pseudo-periodic map would well be called *chiral* if either it is periodic or all of its screw numbers are of the same sign. A chiral pseudo-periodic map is a pseudo-periodic map of negative twist or of positive twist. If a pseudo-periodic map has both positive and negative screw numbers, it will be called *achiral*. In Part I of this memoir, we confined ourselves to chiral pseudo-periodic maps (of negative twist). From the viewpoint of surface topology, it would be more natural to treat not only chiral pseudo-periodic maps but also achiral ones, of course. We tried such a general treatment for some time. However, to construct a generalized quotient for an achiral pseudo-periodic map, we are forced to adopt an artificial convention on signs of intersections between the components consisting of a *tail*-part of the generalized quotient, and we lose the *natural* uniqueness of the generalized quotient of a pseudo-periodic map. Moreover, our construction of generalized quotients is intended to be applied to topology of degeneration of Riemann surfaces in Part II, and for that purpose, we only need chiral pseudo-periodic maps. For these reasons, we gave up our trial to generalize the construction of generalized quotients to achiral maps.

As is immediately seen from the title, the main objects of this memoir are (chiral) pseudo-periodic homeomorphisms and degeneration of Riemann surfaces. Our main point is that these two objects are topologically classified by the same objects, i.e. certain types of "numerical chorizo spaces" together with a cohomology class in the weighted cohomology of their decomposition graphs. This type of chorizo spaces appear as "minimal quotients"of pseudo-periodic homeomorphisms of negative twist, and exactly the same type of chorizo spaces appear also as "normaly minimal

[1] Revised in February 2011.

singular fibers" in one-parameter families of Riemann surfaces. The former objects come from surface topology, while the latter objects come from complex analysis. The authors think interesting that numerical chorizo spaces lie at the common basis of the objects from two different disciplines.

The appearance of pseudo-periodic homeomorphisms of negative twist in degenerating families of Riemann surfaces was clarified through the work of Imayoshi [26], Shiga–H. Tanigawa [58], and finally by Earle and Sipe [19]. We should mention here, however, that the pseudo-periodic nature of the monodromy had been observed for the Milnor fibering [44] at an isolated singular point of a complex hypersurface.

Brieskorn [14] showed, in general dimensions, that the eigenvalues of the (co)homological monodromy are roots of unity. Lê [34] showed, in the case of curves, that the homological monodromy is periodic if the curve is irreducible at the singular point. A'Campo [1] proved that it is not the case if the curve is not irreducible. Also he showed that the geometric (i.e. topological) monodromy is not necessarily periodic, even if the curve is irreducible. A'Campo [2] and Eisenbud – Neumann [20] gave a description of geometric pseudo-periodic monodromies. Finally Lê – Michel – Weber [35] proved that the geometric monodromy is pseudo-periodic ("quasi-finie" in their terminology). Michel – Weber [43] gave a detailed description of the negative twist and showed that the geometric monodromy associated to a complex polynomial map from $\mathbb{C}^2$ to $\mathbb{C}$ (affine case) is also pseudo-periodic of negative twist.

During the two decades, after the completion of our manuscript, several related papers have appeared.

The most related one is, of course, the anouncement of this memoir, which was published in Bull. A.M.S. in 1994 [42]. This might serve as an introduction to this memoir (see also [40]). Pichon [55] used the pseudo-periodicity of the geometric monodromy to characterize the 3-manifolds that appear as the boundary manifolds of degenerating families of Riemann surfaces over a disk. In both of the papers of Pichon [55] and Lê – Michel – Weber [35], Waldhausen's graph manifolds [66, 67] play an important role.

The first authors that put the present memoir to good use were Ashikaga and Ishizaka [7] who gave a complete list of singular fibers in degenerating families of genus 3 (they were more than sixteen hundred!). They very explicitely exploited the algorithm, implicitly contained in the present memoir. It should be noted that the numerical classification of genus 3 singular fibers had been accomplished by Uematsu [64] in 1993 independently of our work.

Xiao and Reid [56] proposed the problem of determining all the "atomic" singular fibers, which are defined as such singular fibers that cannot be "split" by any perturbation of the degenerating families. This problem is very interesting from the viewpoint of the present memoir. By our main result, the topological types of singular fibers are classified by the corresponding topological monodromies around them. Then a natural question to be settled would be if all atomic singular fibers (except for "multiple fibers") correspond to the full (-1)-Dehn twist about a certain simple closed curve. Examples of this geometrical situation are contained

in [3, 29, 38, 39]. (For recent related results, see [5, 6, 8].) Following these trend of ideas, S.Takamura [59, 60] is undertaking a project in solving this problem.

The authors would like to thank Professors D. T. Lê, J.-P. Brasselet, and M. Oka who showed interest in our work, and especially Professor Lê for his explanation on the related results of his own and others. We are also grateful to Professor T. Ashikaga for taking our work seriously and for actively developing related subjects in algebraic geometry and topology, which encouraged us very much. Thanks are also due to Professor Y. Imayoshi for his interest in our results, and for his very benevolent review of our work [28].

In November of 2000 we met in Oberwolfach Professors A'Campo, Weber and Pichon, who encouraged us very strongly to publish our results. We owe to them the final impulse that we needed to conclude the typing of this memoir that has eventually lead to its publication.

Tokyo and Madrid, *Yukio Matsumoto*
September 2009 *José María Montesinos-Amilibia*

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