Lecture Notes in Mathematics 2032

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Algebraic Topology of Finite Topological Spaces and Applications

Springer

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ISBN 978-3-642-22002-9 e-ISBN 978-3-642-22003-6
DOI 10.1007/978-3-642-22003-6
Springer Heidelberg Dordrecht London New York

Lecture Notes in Mathematics ISSN print edition: 0075-8434
ISSN electronic edition: 1617-9692

Library of Congress Control Number: 2011934806

Mathematics Subject Classification (2010): 55-XX; 05-XX; 52-XX; 06-XX; 57-XX

Cover design: deblik, Berlin

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

A las dos bobes

Preface

There should be more math.
This could be mathier.

B.A. Summers

This book is a revised version of my PhD Thesis [5], supervised by Gabriel Minian and defended in March 2009 at the Mathematics Department of the Facultad de Ciencias Exactas y Naturales of the Universidad de Buenos Aires. Some small changes can be found here, following the suggestions of the referees and the editors of the LNM.

Gabriel proposed that we work together in the homotopy theory of finite spaces at the beginning of 2005, claiming that the topic had great potential and could be rich in applications. Very soon I became convinced of this as well. A series of notes by Peter May [51–53] and McCord and Stong's foundational papers [55, 76] were the starting point of our research. May's notes contain very interesting questions and open problems, which motivated the first part of our work.

This presentation of the theory of finite topological spaces includes the most fundamental ideas and results previous to our work and, mainly, our contributions over the last years. It is intended for topologists and combinatorialists, but since it is a self-contained exposition, it is also recommended for advanced undergraduate students and graduate students with a modest knowledge of Algebraic Topology.

The revisions of this book were made during a postdoc at Kungliga Tekniska högskolan, in Stockholm.

Acknowledgements

First and foremost, I would like to express my deepest gratitude to Gabriel Minian, the other person that played a leading role in the creation of this book, from its beginning as advisor of my undergraduate Thesis, until its completion with invaluable suggestions on the final presentation. I thank him for teaching me Algebraic Topology and for sharing with me his expertise. I gratefully acknowledge his encouragement, guidance and his confidence in

me. I thank him for developing with me the ideas presented in this book, and for his unconditional support in each step of my career.

My PhD work at the Universidad de Buenos Aires was supported by a fellowship of CONICET.

I want to thank Anders Björner and the combinatorics group at KTH, where the revisions took place. My postdoc was supported by grant KAW 2005.0098 from the Knut and Alice Wallenberg Foundation.

I am grateful to the anonymous referees and to the editors of the LNM for their suggestions that resulted in a better presentation of my work.

Thanks to my family and friends, especially to my parents and brother. And Mor, and Bubu, of course.

Stockholm,
December 2010

Jonathan A. Barmak

Contents

Introduction

Most of the spaces studied in Algebraic Topology, such as CW-complexes or manifolds, are Hausdorff spaces. In contrast, finite topological spaces are rarely Hausdorff. A topological space with finitely many points, each of which is closed, must be discrete. In some sense, finite spaces are more natural than CW-complexes. They can be described and handled in a combinatorial way because of their strong relationship with finite partially ordered sets, but it is the interaction between their combinatorial and topological structures that makes them important mathematical objects. At first glance, one could think that such non Hausdorff spaces with just a finite number of points are uninteresting, but we will see that the theory of finite spaces can be used to investigate deep well-known problems in Topology, Algebra and Geometry.

In 1937, Alexandroff [1] showed that finite spaces and finite partially ordered sets (posets) are essentially the same objects considered from different points of view. However, it was not until 1966 that strong and deep results on the homotopy theory of finite spaces appeared, shaped in the two foundational and independent papers [76] and [55]. Stong [76] used the combinatorics of finite spaces to explain their homotopy types. This astounding article would have probably gone unnoticed if in the same year, McCord had not discovered the relationship between finite spaces and compact polyhedra. Given a finite topological space X, there exists an associated simplicial complex $\mathcal{K}(X)$ (the order complex) which has the same weak homotopy type as X, and, for each finite simplicial complex K, there is a finite space $\mathcal{X}(K)$ (the face poset) weak homotopy equivalent to K. Therefore, in contrast to what one could have expected at first sight, weak homotopy types of finite spaces coincide with homotopy types of finite CW-complexes. In this way, Stong and McCord put finite spaces in the game, showing implicitly that the interplay between their combinatorics and topology can be used to study homotopy invariants of well-known Hausdorff spaces.

Despite the importance of those papers, finite spaces remained in the shadows for many more years. During that time, the relationship between finite

posets and finite simplicial complexes was exploited, but in most cases without knowing or neglecting the intrinsic topology of the posets. A clear example of this is the 1978 article of Quillen [70], who investigated the connection between algebraic properties of a finite group G and homotopy properties of the simplicial complex associated to the poset $S_p(G)$ of p-subgroups of G. In that beautiful article, Quillen left a challenging conjecture which remains open until this day. Quillen stated the conjecture in terms of the topology of the simplicial complex associated to $S_p(G)$. We will see that the finite space point of view adds a completely new dimension to his conjecture and allows one to attack the problem with new topological and combinatorial tools. We will show that the Whitehead Theorem does not hold for finite spaces: there are weak homotopy equivalent finite spaces with different homotopy types. This distinction between weak homotopy types and homotopy types is lost when we look into the associated polyhedra (because of the Whitehead Theorem) and, in fact, the essence of Quillen's conjecture lies precisely in the distinction between weak homotopy types and homotopy types of finite spaces.

In the last decades, a few interesting papers on finite spaces appeared [35, 65, 77], but the subject certainly did not receive the attention it required. In 2003, Peter May wrote a series of unpublished notes [51–53] in which he synthesized the most important ideas on finite spaces known until that time. In these articles, May also formulated some natural and interesting questions and conjectures which arose from his own research. May was one of the first to note that Stong's combinatorial point of view and the bridge constructed by McCord could be used together to attack problems in Algebraic Topology using finite spaces. My advisor, Gabriel Minian, chose May's notes, jointly with Stong's and McCord's papers, to be the starting point of our research on the Algebraic Topology of Finite Topological Spaces and Applications. This work, based on my PhD Dissertation defended at the Universidad de Buenos Aires in March 2009, is the first detailed exposition on the subject. In these notes I will try to set the basis of the theory of finite spaces, recalling the developments previous to ours, and I will exhibit the most important results of our work in the last years. The concepts and methods that we present in these notes are already being applied by many mathematicians to study problems in different areas.

Many results presented in this work are new and original. Various of them are part of my joint work with Gabriel Minian and appeared in our publications [7–11]. The results on finite spaces previous to ours appear in Chap. 1 and in a few other parts of the book where it is explicitly stated. Chapter 8, on equivariant simple homotopy types and Quillen's conjecture, and Chap. 11, on the Andrews-Curtis conjecture, contain some of the strongest results of these notes. These results are still unpublished.

New homotopical approaches to finite spaces that will not be treated in this book appeared for example in [22, 60] and more categorically oriented in [40, 73]. Applications of our methods and results to graph theory can be

found in [19]. A relationship of finite spaces with toric varieties is discussed in [12].

In the first chapter we recall the correspondence between finite spaces and finite posets, the combinatorial description of homotopy types by Stong and the relationship between weak homotopy types of finite spaces and homotopy types of compact polyhedra found by McCord.

In Chap. 2 we give short basic proofs of many interesting original results. These include: the relationship between homotopy of maps between finite spaces and the discrete notion of homotopy for simplicial maps; an extension of Stong's ideas for pairs of finite spaces; the manifestation of finite homotopy types in the Hausdorff setting; a description of the fundamental group of a finite space; the realization of a finite group as the automorphism group of a finite space and classical constructions in the finite context, including a finite version of the mapping cylinder.

McCord found in [55] a *finite model* of the n-sphere S^n (i.e. a finite space weak homotopy equivalent to S^n) with only $2n+2$ points. May conjectured in his notes that this space is, in our language, a *minimal finite model* of S^n, that is to say a finite model with minimum cardinality. In Chap. 3 we prove that May's conjecture is true. Moreover, the minimal finite model of S^n is unique up to homeomorphism (see Theorem 3.2.2). In this chapter we also study minimal finite models of finite graphs (CW-complexes of dimension 1) and give a full description of them in Theorem 3.3.7. In this case the uniqueness of the minimal finite models depends on the graph. The reason for studying finite models of spaces instead of finite spaces with the same homotopy type is that homotopy types of finite complexes rarely occur in the setting of finite spaces (see Corollary 2.3.4).

Given a finite space X, there exists a homotopy equivalent finite space X_0 which is T_0. That means that for any two points of X_0 there exists an open set which contains only one of them. Therefore, when studying homotopy types of finite spaces, we can restrict our attention to T_0-spaces.

In [76] Stong defined the notion of *linear* and *colinear points* of finite T_0-spaces, which we call *up beat* and *down beat points* following May's terminology. Stong proved that removing a beat point from a finite space does not affect its homotopy type. Moreover, two finite spaces are homotopy equivalent if and only if it is possible to obtain one from the other just by adding and removing beat points. On the other hand, McCord's results suggest that it is more important to study weak homotopy types of finite spaces than homotopy types. In this direction, we generalize Stong's definition of beat points introducing the notion of *weak point* (see Definition 4.2.2). If one removes a weak point x from a finite space X, the resulting space need not be homotopy equivalent to X, however we prove that in this case the inclusion $X \smallsetminus \{x\} \hookrightarrow X$ is a weak homotopy equivalence. As an application of this result, we exhibit an example (Example 4.2.1) of a finite space which is homotopically trivial, i.e. weak homotopy equivalent to a point, but which

is not contractible. This shows that the Whitehead Theorem does not hold for finite spaces, not even for homotopically trivial spaces.

Osaki proved in [65] that if x is a beat point of a finite space X, there is a simplicial collapse from the associated complex $\mathcal{K}(X)$ to $\mathcal{K}(X \smallsetminus \{x\})$. In particular, if two finite spaces are homotopy equivalent, their associated complexes have the same simple homotopy type. However, we noticed that the converse is not true. There are easy examples of non-homotopy equivalent finite spaces with simple homotopy equivalent associated complexes. The removing of beat points is a fundamental move in finite spaces, which gives rise to homotopy types. We asked whether there exists another kind of fundamental move in finite spaces, which corresponds exactly to the simple homotopy types of complexes. We proved that the answer to this question lies exactly in the notion of weak point. We say that there is a *collapse* from a finite space X to a subspace Y if we can obtain Y from X by removing weak points, and we say that two finite spaces have the same *simple homotopy type* if we can obtain one from the other by adding and removing weak points. We denote the first case with $X \searrow Y$ and the second case with $X \diagup\!\!\searrow Y$. The following result, which appears in Chap. 4, says that simple homotopy types of finite spaces correspond precisely to simple homotopy types of the associated complexes.

Theorem 4.2.11.

(a) Let X and Y be finite T_0-spaces. Then, X and Y are simple homotopy equivalent if and only if $\mathcal{K}(X)$ and $\mathcal{K}(Y)$ have the same simple homotopy type. Moreover, if $X \searrow Y$ then $\mathcal{K}(X) \searrow \mathcal{K}(Y)$.
(b) Let K and L be finite simplicial complexes. Then, K and L are simple homotopy equivalent if and only if $\mathcal{X}(K)$ and $\mathcal{X}(L)$ have the same simple homotopy type. Moreover, if $K \searrow L$ then $\mathcal{X}(K) \searrow \mathcal{X}(L)$.

This result allows one to use finite spaces to study problems of classical simple homotopy theory. Indeed, we will use it to study the Andrews-Curtis conjecture and we will use an equivariant version to investigate Quillen's conjecture on the poset of p-subgroups of a finite group.

It is relatively easy to know whether two finite spaces are homotopy equivalent using Stong's ideas, however it is very difficult (algorithmically undecidable in fact) to distinguish if two finite spaces have the same weak homotopy type. Note that this is as hard as recognizing if the associated polyhedra have the same homotopy type. Our results on simple homotopy types provide a first approach in this direction. If two finite spaces have trivial Whitehead group, then they are weak homotopy equivalent if and only if they are simple homotopy equivalent. In particular, a finite space X is homotopically trivial if and only if it is possible to add and remove weak points from X to obtain the singleton $*$. The importance of recognizing homotopically trivial spaces will be evident when we study the conjecture of Quillen. Note that the fundamental move in finite spaces induced by weak

points is easier to handle and describe than the simplicial one because it consists of removing just one single point of the space.

In the fourth section of Chap. 4 we study an analogue of Theorem 4.2.11 for simple homotopy equivalences. We give a description of the maps between finite spaces which correspond to simple homotopy equivalences at the level of complexes. The main result of this section is Theorem 4.4.12. In contrast to the classical situation where simple homotopy equivalences are particular cases of homotopy equivalences, homotopy equivalences between finite spaces are a special kind of simple homotopy equivalences.

As an interesting application of our methods on simple homotopy types, we will prove the following simple homotopy version of Quillen's famous Theorem A.

Theorem 4.5.2. *Let $\varphi : K \to L$ be a simplicial map between finite simplicial complexes. If $\varphi^{-1}(\sigma)$ is collapsible for every simplex σ of L, then $|\varphi|$ is a simple homotopy equivalence.*

In Chap. 5 we study the relationship between homotopy equivalent finite spaces and the associated complexes. The concept of contiguity classes of simplicial maps leads to the notion of *strong homotopy equivalence* (Definition 5.1.4) and *strong homotopy types* of simplicial complexes. This equivalence relation is generated by *strong collapses* which are more restrictive than the usual simplicial collapses. We prove the following result.

Theorem 5.2.1.

(a) If two finite T_0-spaces are homotopy equivalent, their associated complexes have the same strong homotopy type.
(b) If two finite complexes have the same strong homotopy type, the associated finite spaces are homotopy equivalent.

The notion of strong collapsibility is used to study the relationship between the contractibility of a finite space and that of its barycentric subdivision. This concept can be characterized using the nerve of a complex.

The fundamental moves described by beat or weak points are what we call *methods of reduction*. A reduction method is a technique that allows one to change a finite space to obtain a smaller one, preserving some topological properties, such as homotopy type, simple homotopy type, weak homotopy type or the homology groups. In [65], Osaki introduced two methods of this kind which preserve the weak homotopy type, and he asked whether these moves are effective in the following sense: given a finite space X, is it always possible to obtain a minimal finite model of X by applying repeatedly these methods? In Chap. 6 we give an example to show that the answer to this question is negative. In fact, it is a very difficult problem to find minimal finite models of spaces since this question is directly related to the problem of distinguishing weak homotopy equivalent spaces. Moreover, we prove that Osaki's methods of reduction preserve the simple homotopy type. In this

chapter we also study *one-point reduction methods* which consist of removing just one point of the space. For instance, beat points and weak points lead to one-point methods of reduction. In the second section of that chapter, we define the notion of *γ-point* which generalizes the concept of weak point and provides a more applicable method which preserves the weak homotopy type. The importance of this new method is that it is almost the most general possible one-point reduction method. More specifically, we prove the following result.

Theorem 6.2.5. *Let X be a finite T_0-space, and $x \in X$ a point which is neither maximal nor minimal and such that $X \smallsetminus \{x\} \hookrightarrow X$ is a weak homotopy equivalence. Then x is a γ-point.*

In some sense, one-point methods are not sufficient to describe weak homotopy types of finite spaces. Concretely, if $x \in X$ is such that the inclusion $X \smallsetminus \{x\} \hookrightarrow X$ is a weak homotopy equivalence, then $X \smallsetminus \{x\} \diagup\!\!\!\searrow X$ (see Theorem 6.2.8). Therefore, these methods cannot be used to obtain weak homotopy equivalent spaces which are not simple homotopy equivalent.

Another of the problems originally stated by May in [52] consists in extending McCord's ideas in order to model, with finite spaces, not only simplicial complexes, but general CW-complexes. We give an approach to this question in Chap. 7, where the notion of *h-regular CW-complex* is defined. It was already known that regular CW-complexes could be modeled by their face posets. The class of h-regular complexes extends considerably the class of regular complexes and we explicitly construct for each h-regular complex K, a weak homotopy equivalence $K \to \mathcal{X}(K)$. Our results on h-regular complexes allow the construction of new interesting examples of finite models. We also apply these results to investigate quotients of finite spaces and derive a long exact sequence of reduced homology for finite spaces.

Given a finite group G and a prime integer p, we denote by $S_p(G)$ the poset of nontrivial p-subgroups of G. In [70], Quillen proved that if G has a nontrivial normal p-subgroup, then $\mathcal{K}(S_p(G))$ is contractible and he conjectured the converse: if the complex $\mathcal{K}(S_p(G))$ is contractible, G has a nontrivial p-subgroup. Quillen himself proved the conjecture for the case of solvable groups, but the general problem still remains open. Some important advances were achieved in [3]. As we said above, Quillen never considered $S_p(G)$ as a topological space. In 1984, Stong [77] published a second article on finite spaces. He proved some results on the equivariant homotopy theory of finite spaces, which he used to attack Quillen's conjecture. He showed that G has a nontrivial normal p-subgroup if and only if $S_p(G)$ is a contractible finite space. Therefore, the conjecture can be restated in terms of finite spaces as follows: $S_p(G)$ is contractible if and only if it is homotopically trivial. In Chap. 8 we study an equivariant version of simple homotopy types of simplicial complexes and finite spaces and we prove an analogue of Theorem 4.2.11 in this case. Using this result we obtain some new formulations of the conjecture, which

are exclusively written in terms of simplicial complexes. One of these versions states that $\mathcal{K}(S_p(G))$ is contractible if and only if it has trivial equivariant simple homotopy type. We also obtain formulations of the conjecture in terms of the polyhedron associated to the much smaller poset $A_p(G)$ of the elementary abelian p-subgroups.

In Chap. 9 we describe homotopy properties of the so called *reduced lattices*, which are finite lattices with their top and bottom elements removed. We also introduce the $\mathcal{L}$ construction, which associates a new simplicial complex to a given finite space. This application, which is closed related to the nerve of a complex, was not included originally in my Dissertation [5]. We compare the homotopy type of a finite T_0-space X and the strong homotopy type of $\mathcal{L}(X)$. At the end of the chapter, another restatement of Quillen's conjecture is given using the complex $L_p(G) = \mathcal{L}(S_p(G))$. This version of the conjecture is closely related to the so called Evasiveness conjecture.

Chapter 10 is devoted to the study of fixed point sets of maps. We study the relationship between the fixed points of a simplicial automorphism and the fixed points of the associated map between finite spaces. We use this result to prove a stronger version of Lefschetz Theorem for simplicial automorphisms.

In the last chapter of these notes we exhibit some advances concerning the Andrews-Curtis conjecture. The geometric Andrews-Curtis conjecture states that if K is a contractible complex of dimension 2, then it 3-deforms to a point, i.e. it can be deformed into a point by a sequence of collapses and expansions which involve complexes of dimension not greater than 3. This long standing problem stated in the sixties, is closely related to Zeeman's conjecture and hence, to the famous Poincaré conjecture. With the proof of the Poincaré conjecture by Perelman, and by [30], we know now that the geometric Andrews-Curtis conjecture is true for *standard spines* (see [72]), but it still remains open for general 2-complexes. Inspired by our results on simple homotopy theory of finite spaces and simplicial complexes, we define the notion of *quasi constructible* 2-*complexes* which generalizes the concept of constructible complexes. Using techniques of finite spaces we prove that contractible quasi constructible 2-complexes 3-deform to a point. In this way we substantially enlarge the class of complexes which are known to satisfy the conjecture.

Throughout the book, basic results of Algebraic Topology will be assumed to be known by the reader. Nevertheless, we have included an appendix at the end of the notes, where we recall some basic concepts, ideas and classical results on simplicial complexes and CW-complexes that might be useful to the non-specialist.