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# From Objects to Diagrams for Ranges of Functors

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# Foreword

The aim of the present work is to introduce a general method, applicable to various fields of mathematics, that enables us to gather information on the range of a functor  $\Phi$ , thus making it possible to solve previously intractable representation problems with respect to  $\Phi$ . This method is especially effective in case the problems in question are “cardinality-sensitive”, that is, an analogue of the cardinality function turns out to play a crucial role in the description of the members of the range of  $\Phi$ .

Let us first give a few examples of such problems. The first three belong to the field of universal algebra, the fourth to the field of ring theory (nonstable K-theory of rings).

*Context 1.* The classical Grätzer–Schmidt Theorem, in universal algebra, states that every  $(\vee, 0)$ -semilattice is isomorphic to the compact congruence lattice of some algebra. Can this result be extended to *diagrams* of  $(\vee, 0)$ -semilattices?

*Context 2.* For a member  $\mathbf{A}$  of a quasivariety  $\mathcal{A}$  of algebraic systems, we denote by  $\text{Con}_c^{\mathcal{A}} \mathbf{A}$  the  $(\vee, 0)$ -semilattice of all compact elements of the lattice of all congruences of  $\mathbf{A}$  with quotient in  $\mathcal{A}$ ; further, we denote by  $\text{Con}_{c,r} \mathcal{A}$  the class of all isomorphic copies of  $\text{Con}_c^{\mathcal{A}} \mathbf{A}$  where  $\mathbf{A} \in \mathcal{A}$ . For quasivarieties  $\mathcal{A}$  and  $\mathcal{B}$  of algebraic systems, we denote by  $\text{crit}_r(\mathcal{A}; \mathcal{B})$  (*relative critical point* between  $\mathcal{A}$  and  $\mathcal{B}$ ) the least possible cardinality, if it exists, of a member of  $(\text{Con}_{c,r} \mathcal{A}) \setminus (\text{Con}_{c,r} \mathcal{B})$ , and  $\infty$  otherwise. What are the possible values of  $\text{crit}_r(\mathcal{A}; \mathcal{B})$ , say for  $\mathcal{A}$  and  $\mathcal{B}$  both with finite language?

*Context 3.* Let  $\mathcal{V}$  be a nondistributive variety of lattices and let  $F$  be the free lattice in  $\mathcal{V}$  on  $\aleph_1$  generators. Does  $F$  have a congruence-permutable, congruence-preserving extension?

*Context 4.* Let  $E$  be an exchange ring. Is there a (von Neumann) regular ring, or a  $C^*$ -algebra of real rank zero,  $R$  with the same nonstable K-theory as  $E$ ?

It turns out that each of these problems can be reduced to a category-theoretical problem of the following general kind.

Let  $\mathcal{A}, \mathcal{B}, \mathcal{S}$  be categories, let  $\Phi: \mathcal{A} \rightarrow \mathcal{S}$  and  $\Psi: \mathcal{B} \rightarrow \mathcal{S}$  be functors. We are also given a subcategory  $\mathcal{S}^{\Rightarrow}$  of  $\mathcal{S}$ , of which the arrows will be called *double arrows* and written  $f: X \Rightarrow Y$ . We assume that for “many” objects  $A$  of  $\mathcal{A}$ , there are an object  $B$  of  $\mathcal{B}$  and a double arrow  $\chi: \Psi(B) \Rightarrow \Phi(A)$ . We also need to assume that our categorical data forms a so-called *larder*. In such a case, we establish that under certain combinatorial assumptions on a poset  $P$ , for “many” diagrams  $\vec{A} = (A_p, \alpha_p^q \mid p \leq q \text{ in } P)$  from  $\mathcal{A}$ , a similar conclusion holds at the diagram  $\Phi \vec{A}$ , that is, there are a  $P$ -indexed diagram  $\vec{B}$  from  $\mathcal{B}$  and a double arrow  $\vec{\chi}: \Psi \vec{B} \Rightarrow \Phi \vec{A}$  from  $\mathcal{S}^P$ . The combinatorial assumptions on  $P$  imply that every principal ideal of  $P$  is a join-semilattice and the set of all upper bounds of any finite subset is a finitely generated upper subset.

We argue by concentrating all the relevant properties of the diagram  $\vec{A}$  into a *condensate* of  $\vec{A}$ , which is a special kind of directed colimit of finite products of the  $A_p$  for  $p \in P$ . Our main result, the *Condensate Lifting Lemma* (CLL), reduces the liftability of a diagram to the liftability of a condensate, modulo a list of elementary verifications of categorical nature. The impact of CLL on the four problems above can be summarized as follows:

*Context 1.* The Grätzer–Schmidt Theorem can be extended to any diagram of  $(\vee, 0)$ -semilattices and  $(\vee, 0)$ -homomorphisms indexed by a finite poset (resp., assuming a proper class of Erdős cardinals, an arbitrary poset), lifting with algebras of variable similarity type.

*Context 2.* We prove that in a host of situations, either  $\text{crit}_r(\mathcal{A}; \mathcal{B}) < \aleph_\omega$  or  $\text{crit}_r(\mathcal{A}; \mathcal{B}) = \infty$ . This holds, in particular, if  $\mathcal{A}$  is a locally finite quasivariety with finitely many relations while  $\mathcal{B}$  is a finitely generated, congruence-modular variety with finite similarity type of algebras of finite type (e.g., groups, lattices, modules over a finite ring).

*Context 3.* The free  $\mathcal{V}$ -lattice  $F$  has no congruence-permutable, congruence-preserving extension in any similarity type containing the lattice type. Due to earlier work by Grätzer et al. [30], if  $\mathcal{V}$  is locally finite, then the cardinality  $\aleph_1$  is optimal.

*Context 4.* By using the results of the present work, the second author proves in Wehrung [71] that the answer is no (both for regular rings and for  $C^*$ -algebras of real rank zero), with a counterexample of cardinality  $\aleph_3$ .

We also pave the way to solutions of further beforehand intractable open problems:

- The determination of all the possible critical points between finitely generated varieties of lattices [20], then between varieties  $\mathcal{A}$  and  $\mathcal{B}$  of algebras such  $\mathcal{A}$  is locally finite while  $\mathcal{B}$  is finitely generated with finite similarity type and omits tame congruence theory types **1** and **5** [22].

- The problem whether every lattice of cardinality  $\aleph_1$ , in the variety generated by  $\mathbf{M}_3$ , has a congruence  $m$ -permutable, congruence-preserving extension for some positive integer  $m$  [21].
- A 1962 problem by Jónsson about coordinatizability of sectionally complemented modular lattices without unit [69].





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