Lecture Notes in Mathematics 2029

For further volumes:
http://www.springer.com/series/304

Pierre Gillibert • Friedrich Wehrung

From Objects to Diagrams for Ranges of Functors

Springer

Pierre Gillibert
Charles University in Prague
Department of Mathematics
Sokolovská 83
18600 Prague Praha
Czech Republic
pgillibert@yahoo.fr

Friedrich Wehrung
University of Caen, LMNO, CNRS
UMR 6139,
Department of Mathematics
14032 Caen, Cedex
France
wehrung@math.unicaen.fr

ISBN 978-3-642-21773-9 e-ISBN 978-3-642-21774-6
DOI 10.1007/978-3-642-21774-6
Springer Heidelberg Dordrecht London New York

Lecture Notes in Mathematics ISSN print edition: 0075-8434
ISSN electronic edition: 1617-9692

Library of Congress Control Number: 2011933135

Mathematics Subject Classification (2010): Primary 18A30, 18A25, 18A20, 18A35;
Secondary 03E05, 05D10, 06A07, 06A12

Cover design: deblik, Berlin

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Foreword

The aim of the present work is to introduce a general method, applicable to various fields of mathematics, that enables us to gather information on the range of a functor Φ, thus making it possible to solve previously intractable representation problems with respect to Φ. This method is especially effective in case the problems in question are "cardinality-sensitive", that is, an analogue of the cardinality function turns out to play a crucial role in the description of the members of the range of Φ.

Let us first give a few examples of such problems. The first three belong to the field of universal algebra, the fourth to the field of ring theory (nonstable K-theory of rings).

Context 1. The classical Grätzer–Schmidt Theorem, in universal algebra, states that every $(\vee, 0)$-semilattice is isomorphic to the compact congruence lattice of some algebra. Can this result be extended to *diagrams* of $(\vee, 0)$-semilattices?

Context 2. For a member $\boldsymbol{A}$ of a quasivariety $\mathcal{A}$ of algebraic systems, we denote by $\mathrm{Con}_{\mathrm{c}}^{\mathcal{A}}\,\boldsymbol{A}$ the $(\vee, 0)$-semilattice of all compact elements of the lattice of all congruences of $\boldsymbol{A}$ with quotient in $\mathcal{A}$; further, we denote by $\mathrm{Con}_{\mathrm{c,r}}\,\mathcal{A}$ the class of all isomorphic copies of $\mathrm{Con}_{\mathrm{c}}^{\mathcal{A}}\,\boldsymbol{A}$ where $\boldsymbol{A} \in \mathcal{A}$. For quasivarieties $\mathcal{A}$ and $\mathcal{B}$ of algebraic systems, we denote by $\mathrm{crit}_{\mathrm{r}}(\mathcal{A}; \mathcal{B})$ (*relative critical point* between $\mathcal{A}$ and $\mathcal{B}$) the least possible cardinality, if it exists, of a member of $(\mathrm{Con}_{\mathrm{c,r}}\,\mathcal{A}) \setminus (\mathrm{Con}_{\mathrm{c,r}}\,\mathcal{B})$, and ∞ otherwise. What are the possible values of $\mathrm{crit}_{\mathrm{r}}(\mathcal{A}; \mathcal{B})$, say for $\mathcal{A}$ and $\mathcal{B}$ both with finite language?

Context 3. Let $\mathcal{V}$ be a nondistributive variety of lattices and let F be the free lattice in $\mathcal{V}$ on $\aleph_1$ generators. Does F have a congruence-permutable, congruence-preserving extension?

Context 4. Let E be an exchange ring. Is there a (von Neumann) regular ring, or a C*-algebra of real rank zero, R with the same nonstable K-theory as E?

It turns out that each of these problems can be reduced to a category-theoretical problem of the following general kind.

Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$ be categories, let $\Phi\colon \mathcal{A} \to \mathcal{S}$ and $\Psi\colon \mathcal{B} \to \mathcal{S}$ be functors. We are also given a subcategory $\mathcal{S}^{\Rightarrow}$ of $\mathcal{S}$, of which the arrows will be called *double arrows* and written $f\colon X \Rightarrow Y$. We assume that for "many" *objects* A of $\mathcal{A}$, there are an object B of $\mathcal{B}$ and a double arrow $\chi\colon \Psi(B) \Rightarrow \Phi(A)$. We also need to assume that our categorical data forms a so-called *larder*. In such a case, we establish that under certain combinatorial assumptions on a poset P, for "many" *diagrams* $\overrightarrow{A} = \bigl(A_p, \alpha_p^q \mid p \leq q \text{ in } P\bigr)$ from $\mathcal{A}$, a similar conclusion holds at the diagram $\Phi\overrightarrow{A}$, that is, there are a P-indexed diagram $\overrightarrow{B}$ from $\mathcal{B}$ and a double arrow $\overrightarrow{\chi}\colon \Psi\overrightarrow{B} \Rightarrow \Phi\overrightarrow{A}$ from $\mathcal{S}^P$. The combinatorial assumptions on P imply that every principal ideal of P is a join-semilattice and the set of all upper bounds of any finite subset is a finitely generated upper subset.

We argue by concentrating all the relevant properties of the diagram $\overrightarrow{A}$ into a *condensate* of $\overrightarrow{A}$, which is a special kind of directed colimit of finite products of the A_p for $p \in P$. Our main result, the *Condensate Lifting Lemma* (CLL), reduces the liftability of a diagram to the liftability of a condensate, modulo a list of elementary verifications of categorical nature. The impact of CLL on the four problems above can be summarized as follows:

Context 1. The Grätzer–Schmidt Theorem can be extended to any diagram of $(\vee, 0)$-semilattices and $(\vee, 0)$-homomorphisms indexed by a finite poset (resp., assuming a proper class of Erdős cardinals, an arbitrary poset), lifting with algebras of variable similarity type.

Context 2. We prove that in a host of situations, either $\operatorname{crit}_{\mathrm{r}}(\mathcal{A};\mathcal{B}) < \aleph_\omega$ or $\operatorname{crit}_{\mathrm{r}}(\mathcal{A};\mathcal{B}) = \infty$. This holds, in particular, if $\mathcal{A}$ is a locally finite quasivariety with finitely many relations while $\mathcal{B}$ is a finitely generated, congruence-modular variety with finite similarity type of algebras of finite type (e.g., *groups*, *lattices*, *modules over a finite ring*).

Context 3. The free $\mathcal{V}$-lattice F has no congruence-permutable, congruence-preserving extension in any similarity type containing the lattice type. Due to earlier work by Grätzer et al. [30], if $\mathcal{V}$ is locally finite, then the cardinality $\aleph_1$ is optimal.

Context 4. By using the results of the present work, the second author proves in Wehrung [71] that the answer is no (both for regular rings and for C*-algebras of real rank zero), with a counterexample of cardinality $\aleph_3$.

We also pave the way to solutions of further beforehand intractable open problems:

- The determination of all the possible critical points between finitely generated varieties of lattices [20], then between varieties $\mathcal{A}$ and $\mathcal{B}$ of algebras such $\mathcal{A}$ is locally finite while $\mathcal{B}$ is finitely generated with finite similarity type and omits tame congruence theory types **1** and **5** [22].

- The problem whether every lattice of cardinality $\aleph_1$, in the variety generated by $\boldsymbol{M}_3$, has a congruence m-permutable, congruence-preserving extension for some positive integer m [21].
- A 1962 problem by Jónsson about coordinatizability of sectionally complemented modular lattices without unit [69].

Contents