

Lecture Notes in Mathematics

2026

Editors:

J.-M. Morel, Cachan

B. Teissier, Paris

Subseries:

École d'Été de Probabilités de Saint-Flour

For further volumes:

<http://www.springer.com/series/304>

Saint-Flour Probability Summer School



The Saint-Flour volumes are reflections of the courses given at the Saint-Flour Probability Summer School. Founded in 1971, this school is organised every year by the Laboratoire de Mathématiques (CNRS and Université Blaise Pascal, Clermont-Ferrand, France). It is intended for PhD students, teachers and researchers who are interested in probability theory, statistics, and in their applications.

The duration of each school is 13 days (it was 17 days up to 2005), and up to 70 participants can attend it. The aim is to provide, in three high-level courses, a comprehensive study of some fields in probability theory or Statistics. The lecturers are chosen by an international scientific board. The participants themselves also have the opportunity to give short lectures about their research work.

Participants are lodged and work in the same building, a former seminary built in the 18th century in the city of Saint-Flour, at an altitude of 900 m. The pleasant surroundings facilitate scientific discussion and exchange.

The Saint-Flour Probability Summer School is supported by:

- Université Blaise Pascal
- Centre National de la Recherche Scientifique (C.N.R.S.)
- Ministère délégué à l'Enseignement supérieur et à la Recherche

For more information, see back pages of the book and
<http://math.univ-bpclermont.fr/stflour/>

Jean Picard
Summer School Chairman
Laboratoire de Mathématiques
Université Blaise Pascal
63177 Aubière Cedex
France

Yves Le Jan

Markov Paths, Loops and Fields

École d'Été de Probabilités
de Saint-Flour XXXVIII-2008

Yves Le Jan
Université Paris-Sud
Département de Mathématiques
Bât.425
91405 Orsay Cedex
France
yves.lejan@math.upsud.fr

ISBN 978-3-642-21215-4 e-ISBN 978-3-642-21216-1
DOI 10.1007/978-3-642-21216-1
Springer Heidelberg Dordrecht London New York

Lecture Notes in Mathematics ISSN print edition: 0075-8434
ISSN electronic edition: 1617-9692

Library of Congress Control Number: 2011932434

Mathematics Subject Classification (2011): Primary 60J27, 60K35; Secondary 60J45

© Springer-Verlag Berlin Heidelberg 2011

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilm or in any other way, and storage in data banks. Duplication of this publication or parts thereof is permitted only under the provisions of the German Copyright Law of September 9, 1965, in its current version, and permission for use must always be obtained from Springer. Violations are liable to prosecution under the German Copyright Law.

The use of general descriptive names, registered names, trademarks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

Cover design: deblik, Berlin

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Preface

The purpose of these notes is to explore some simple relations between Markovian path and loop measures, the Poissonian ensembles of loops they determine, their occupation fields, uniform spanning trees, determinants, and Gaussian Markov fields such as the free field. These relations are first studied in complete generality in the finite discrete setting, then partly generalized to specific examples in infinite and continuous spaces.

These notes contain the results published in [27] where the main emphasis was put on the study of occupation fields defined by Poissonian ensembles of Markov loops. These were defined in [18] for planar Brownian motion in relation with SLE processes and in [19] for simple random walks. They appeared informally already in [52]. For half integral values $\frac{k}{2}$ of the intensity parameter α , these occupation fields can be identified with the sum of squares of k copies of the associated free field (i.e. the Gaussian field whose covariance is given by the Green function). This is related to Dynkin's isomorphism (cf. [6, 23, 33]).

As in [27], we first present the theory in the elementary framework of symmetric Markov chains on a finite space. After some generalities on graphs and symmetric Markov chains, we study the σ -finite loop measure associated to a field of conductances. Then we study geodesic loops with an exposition of results of independent interest, such as the calculation of Ihara's zeta function. After that, we turn our attention to the Poisson process of loops and its occupation field, proving also several other interesting results such as the relation between loop ensembles and spanning trees given by Wilson algorithm and the reflection positivity property. Spanning trees are related to the fermionic Fock space as Markovian loop ensembles are related to the bosonic Fock space, represented by the free field. We also study the decompositions of the loop ensemble induced by the excursions into the complement of any given set.

Then we show that some results can be extended to more general Markov processes defined on continuous spaces. There are no essential difficulties for the occupation field when points are not polar but other cases are

more problematic. As for the square of the free field, cases for which the Green function is Hilbert Schmidt such as those corresponding to two and three dimensional Brownian motion can be dealt with through appropriate renormalization.

We show that the renormalized powers of the occupation field (i.e. the self intersection local times of the loop ensemble) converge in the case of the two dimensional Brownian motion and that they can be identified with higher even Wick powers of the free field when α is a half integer.

At first, we suggest the reader could omit a few sections which are not essential for the understanding of the main results. These are essentially some of the generalities on graphs, results about wreath products, infinite discrete graphs, boundaries, zeta functions, geodesics and geodesic loops. The section on reflexion positivity, and, to a lesser extent, the one on decompositions are not central. The last section on continuous spaces is not written in full detail and may seem difficult to the least experienced readers.

These notes include those of the lecture I gave in St Flour in July 2008 with some additional material. I choose this opportunity to express my thanks to Jean Picard, to the audience and to the readers of the preliminary versions whose suggestions were very useful, in particular to Juergen Angst, Cedric Bordenave, Cedric Boutiller, Jinshan Chang, Antoine Dahlqvist, Thomas Duquesne, Michel Emery, Jacques Franchi, Hatem Hajri, Liza Jones, Adrien Kassel, Rick Kenyon, Sophie Lemaire, Thierry Levy, Titus Lupu, Gregorio Moreno, Jay Rosen (who pointed out a mistake in the expression of renormalization polynomials), Bruno Shapira, Alain Sznitman, Vincent Vigon, Lorenzo Zambotti and Jean Claude Zambrini.

Contents

1	Symmetric Markov Processes on Finite Spaces	1
1.1	Graphs	1
1.2	Energy	5
1.3	Feynman–Kac Formula	7
1.4	Recurrent Extension of a Transient Chain	8
1.5	Transfer Matrix	10
2	Loop Measures	13
2.1	A Measure on Based Loops	13
2.2	First Properties	15
2.3	Loops and Pointed Loops	16
2.4	Occupation Field	19
2.5	Wreath Products	25
2.6	Countable Spaces	26
2.7	Zeta Functions for Discrete Loops	27
3	Geodesic Loops	29
3.1	Reduction	29
3.2	Geodesic Loops and Conjugacy Classes	29
3.3	Geodesics and Boundary	30
3.4	Closed Geodesics and Associated Zeta Function	32
4	Poisson Process of Loops	35
4.1	Definition	35
4.2	Moments and Polynomials of the Occupation Field	39
4.3	Hitting Probabilities	43
5	The Gaussian Free Field	47
5.1	Dynkin’s Isomorphism	47
5.2	Wick Products	49

5.3	The Gaussian Fock Space Structure.....	51
5.4	The Poissonian Fock Space Structure.....	53
6	Energy Variation and Representations	57
6.1	Variation of the Energy Form	57
6.2	One-Forms and Representations.....	60
7	Decompositions	65
7.1	Traces of Markov Chains and Energy Decomposition	65
7.2	Excursion Theory.....	67
7.2.1	The One Point Case and the Excursion Measure ...	68
7.3	Conditional Expectations	70
7.4	Branching Processes with Immigration	71
7.5	Another Expression for Loop Hitting Distributions.....	73
8	Loop Erasure and Spanning Trees	75
8.1	Loop Erasure.....	75
8.2	Wilson Algorithm.....	78
8.3	The Transfer Current Theorem.....	82
8.4	The Skew-Symmetric Fock Space	86
9	Reflection Positivity	91
9.1	Main Result	91
9.2	A Counter Example	95
9.3	Physical Hilbert Space and Time Shift	96
10	The Case of General Symmetric Markov Processes	99
10.1	Overview	99
10.1.1	Loop Hitting Distributions.....	99
10.1.2	Determinantal Processes	100
10.1.3	Occupation Field and Continuous Branching	100
10.1.4	Generalized Fields and Renormalization.....	101
10.2	Isomorphism for the Renormalized Occupation Field.....	101
10.3	Renormalized Powers	106
10.3.1	Final Remarks	112
	References	115
	Index	119
	Programme of the school	121
	List of participants	123