

# Lecture Notes in Mathematics

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# Disorder and Critical Phenomena Through Basic Probability Models

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# Preface

These notes are a revised version of the ones that I have prepared and used for my course at the 40th Saint-Flour Probability Summer School. I was extremely happy to receive the invitation and giving the course has been a real pleasure. This marks my my third time participating in the school and this preface is the occasion to compliment Jean Picard for his discrete, smooth and efficient way of running it.

This invitation gave me the opportunity to rethink my research activities in the last 5 or 6 years, trying to take a somewhat different standpoint: in the end I realized that I was just trying to go back to the original motivations. In this period in fact I have been working mostly on localization phenomena in certain disordered systems and a class of models – the pinning models – took a leading position. But the question driving my interest was and is: what is the effect of disorder on phase transitions and on critical phenomena? So the result is that, if we look at these lectures from a technical viewpoint, they are about the specific class of statistical mechanics models that I call disordered pinning models. For this class we do have fairly satisfactory answers: essentially all the physical predictions on which there was a general consensus have now been established on firm mathematical grounds and there are now also rigorous result about some controversial physical statements. But, beyond the purely technical aspects, these notes are also an invitation to look beyond pinning models, that is, to more general statistical mechanics models.

It suffices to browse through these pages to realize that “more general statistical mechanics models” essentially reduces to the Ising model (and this is still, definitely, too much for these notes). The Ising model is going to accompany us along the various steps, but (hopefully!) in a way that it is not too invasive: the reader who is only interested in pinning phenomena should be able to follow leaving aside the sections on the Ising model, in which the presentation is rather informal. The choice of keeping disordered Ising models issues at an informal level is also due to the lack of rigorous results, in spite of some absolutely impressive achievements, and these portions of the notes present a number of open problems, which are most probably really challenging.

I would like to stress that since (polymer, interface, Markov process, etc.) pinning models are the means and not the aim, the modeling aspects and the very many variations or closely related classes of models are reduced to remarks or are even neglected entirely unless directly related to the main line of the notes. In this sense these notes do not review, for example, the vast literature on polymer models, not even that on general pinning or localization phenomena.

Moreover, these notes do not include hierarchical models on diamond lattices. Choices had to be made and this was the most painful one for at least two reasons: on one hand, part of the results and of the phenomena that I present have been obtained or have been understood first in the hierarchical set-up and, on the other, these somewhat exotic models definitely have a particular inner beauty.

I am presenting the combined work of several persons, to whom I am deeply grateful and indebted. I want to especially thank my closest collaborators and the people with whom I have discussed the subject of these notes most: Francesco Caravenna, Hubert Lacoin and Fabio Toninelli. Moreover special thanks are due to Bernard Derrida, who shaped my vision of the Harris criterion and who helped me in going through the physics literature (needless to say, I take full responsibility for what I have written on physics issues and I am absolutely aware that Bernard would have put things differently). I certainly cannot forget that my interest in disordered models and in localization phenomena dates back to my valued interaction, which persisted through the years, with Erwin Bolthausen.

Finally, I would like to give a big “thank you” to Lydia, Micol and Raika for their presence, in Saint-Flour, before and after.

Paris  
March 2011

*Giambattista Giacomin*

# Contents

<b>1</b>	<b>Introduction</b>	1
	References	4
<b>2</b>	<b>Homogeneous Pinning Systems:</b>	
	<b>A Class of Exactly Solved Models</b>	5
2.1	What Happens if We Reward a Random Walk When it Touches the Origin?	5
2.1.1	The Random Walk Pinning Model	5
2.1.2	Visits to the Origin and the Computation of the Partition Function	7
2.1.3	From Partition Function Estimates to Properties of the System	10
2.2	The General Homogeneous Pinning Model	14
2.3	Phase Transition and Critical Behavior	18
2.4	A First Look at a Crucial Notion: The Correlation Length	19
2.5	Why Do People Look at Pinning Models?	
	A Modeling Intermezzo	21
2.5.1	Polymer Pinning by a Defect	21
2.5.2	Interfaces in Two Dimensions	21
2.5.3	DNA Denaturation: The Poland–Scheraga Model	24
2.6	A Look at the Literature	24
	References	26
<b>3</b>	<b>Introduction to Disordered Pinning Models</b>	29
3.1	The Disordered Pinning Model	29
3.2	Super-Additivity, Free Energy, and Localization	32
3.2.1	Two Important Remarks	34
3.3	Self-Averaging Property, Effect of Boundary Condition	35
3.3.1	Proof of Proposition 3.2	35
3.3.2	Free and Constrained Models	37

3.4	A Look at the Literature and, Once Again, Correlation Length Issues.....	38
	References .....	40
<b>4</b>	<b>Irrelevant Disorder Estimates .....</b>	<b>41</b>
4.1	Disorder and Critical Behavior: What to Expect?.....	41
4.1.1	First Approach: An Expansion in Powers of $\beta^2$ .....	42
4.1.2	Second Approach: A 2-Replica Argument .....	44
4.2	Disorder is Irrelevant if $\alpha < 1/2$ (and if $\beta$ is Not Too Large): A Proof .....	46
4.3	A Look at the Literature .....	49
	References .....	50
<b>5</b>	<b>Relevant Disorder Estimates: The Smoothing Phenomenon .....</b>	<b>51</b>
5.1	Smoothing for Gaussian Charges: The Rare Stretch Strategy .....	51
5.2	More General Charge Distributions .....	55
5.3	Back to and Beyond Harris Criterion: Disorder and Smoothing ....	55
5.3.1	Disorder and Phase Transitions .....	56
5.3.2	Harris' Heuristic Argument .....	57
5.3.3	Relevance and Irrelevance .....	58
5.3.4	The Diluted Ising Model.....	58
5.3.5	Random External Fields .....	59
5.4	A Further Look at the Literature .....	60
	References .....	60
<b>6</b>	<b>Critical Point Shift: The Fractional Moment Method .....</b>	<b>63</b>
6.1	Main Result and Overview .....	63
6.2	The Basic Fractional Moment Estimates .....	65
6.3	The $\alpha > 1$ Case .....	67
6.3.1	A Different Look on Proposition 6.3.....	67
6.3.2	A First Coarse Graining Procedure: Iterated Fractional Moment Estimates .....	68
6.3.3	Finite Volume Estimates: The Proof of Theorem 6.1 for $\alpha > 1$ .....	70
6.4	The $\alpha = 1$ Case .....	74
6.5	The $\alpha \in (1/2, 1)$ Case .....	75
6.5.1	Bounds for Correlation Length Size Systems .....	76
6.5.2	Proof of Theorem 6.1, Case $\alpha \in (1/2, 1)$ .....	78
6.6	The $\alpha = 1/2$ Case .....	79
6.6.1	Estimates up to the (Annealed) Correlation Length: Gaussian Case .....	79
6.6.2	Beyond the Correlation Length: The Proof of Theorem 6.1 ( $\alpha = 1/2$ ) .....	83
6.7	A Look at the Literature .....	87
	References .....	88



<b>7 The Coarse Graining Procedure</b>	91
7.1 Coarse Graining Estimates	91
References	99
<b>8 Path Properties</b>	101
8.1 Overview	101
8.2 A Quick Look at Concentration Inequalities	102
8.3 The Localized Regime	104
8.3.1 A Basic Observation (and its Consequences)	104
8.3.2 On $\mu(\beta, h)$ and $F(\beta, h)$	106
8.4 The Delocalized Regime	108
8.5 Path Behavior: Overview of What is Known and What is Not	109
8.5.1 On the Localized (and Critical) Regime	110
8.5.2 On the Delocalized Regime	111
References	111
<b>A Discrete Renewal Theory:</b>	
<b>Basic (and a Few Less Basic) Facts and Estimates</b>	113
A.1 A Crash Course on Renewal Theory	113
A.1.1 Renewal and Markov Chains	113
A.1.2 The Renewal Theorem	114
A.1.3 Beyond the Renewal Theorem	115
A.1.4 Convergence of Renewal and Point Processes	116
A.2 Some Pinning Oriented Renewal Issues	117
A.2.1 On Boundary Effects	117
A.2.2 Two Scaling Results for Renewal Processes	118
A.2.3 On the Derivatives of the Free Energy Near Criticality	122
References	125
<b>Index</b>	127
<b>List of participants</b>	129



# Frequently Used Notations

$\mathbb{N}$	$\{1, 2, 3, \dots\}$
$a \wedge b, a \vee b$	$\min(a, b), \max(a, b)$
$ E , \mathcal{P}(E), \mathbf{1}_E$	Cardinality, set of all subsets, indicator function of $E$
$\{a_n\}_n, \{b_n\}_n, \dots$	Sequences of real numbers
$a_n \sim b_n$	$\lim_n a_n/b_n = 1$
$a_n \approx b_n$	Used when one does not want, or cannot, be precise
$\tau = \{\tau_j\}_{j=0,1,\dots}$	Renewal sequence, often seen as subset of $\mathbb{N} \cup \{0\}$
$S = \{S_j\}_{j=0,1,\dots}$	Random walk
<b>P</b>	Law of $\tau$ or law of $S$ , according to the context
$K(n)$	$\mathbf{P}(\tau_1 = n), n = 1, 2, \dots, \infty$ (Chap. 2, Sect. 2.2)
$\overline{K}(n)$	$\sum_{j>n} K(n) \leq 1$ ( $n = 0, 1, \dots$ , sum does not include $\infty$ )
$\tilde{K}_h(n)$	(2.10) and Chap. 2, Sect. 2.2
$\tilde{\tau}^{(h)}$	Renewal with inter-arrival law $\tilde{K}_h(n)$
$\theta$	Left shift operator: $(\theta a)_n = a_{n+1}$
$h$	Pinning potential
$\delta_n$	$\mathbf{1}_{n \in \tau}$ (abuse of notation for $\mathbf{1}_\tau(n)$ , here $\tau \subset \mathbb{N} \cup \{0\}$ )
$Z_{N,h}, Z_{N,h}^{\mathbf{f}}$	Partition function of homogeneous system (constrained, free)
$\mathbf{P}_{N,h}, \mathbf{P}_{N,h}^{\mathbf{f}}$	Probability law of homogeneous system (constrained, free)
$F(h)$	Free energy of model with homogeneous pinning potential $h$
$\kappa$	correlation length
$\omega, \mathbb{P}$	disorder or charge sequence, law of $\omega$ : Definition 3.1
$\mathbf{M}(\beta)$	$\mathbb{E} \exp(\beta \omega_1)$ : Definition 3.1
$\beta$	disorder strength parameter
$Z_{N,\omega} = Z_{N,\omega,\beta,h}$	Partition function of disordered system (constrained)
$Z_{N,\omega}^{\mathbf{f}} = Z_{N,\omega,\beta,h}^{\mathbf{f}}$	Partition function of disordered system (free)
$\mathbf{P}_{N,\omega} = \mathbf{P}_{N,\omega,\beta,h}$	Disordered system probability (constrained)
$\mathbf{P}_{N,\omega}^{\mathbf{f}} = \mathbf{P}_{N,\omega,\beta,h}^{\mathbf{f}}$	Disordered system probability (free)
$F(\beta, h)$	Free energy of the disordered pinning model ( $F(0, h) = F(h)$ )
<b>IID</b>	Independent and Identically Distributed