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Boris N. Khoromskij
Gabriel Wittum

Numerical Solution of Elliptic Differential Equations by Reduction to the Interface

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Boris N. Khoromskij

Max-Planck-Institut für Mathematik
in den Naturwissenschaften
Inselstraße 22-26
04103 Leipzig, Germany
e-mail: boris.khoromskij@mis.mpg.de

Gabriel Wittum

Universität Heidelberg
Interdisziplinäres Zentrum
für Wissenschaftliches Rechnen (IWR)
Im Neuenheimer Feld 368
69120 Heidelberg, Germany
e-mail: wittum@iwr.uni-heidelberg.de

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Preface

During the last decade essential progress has been achieved in the analysis and implementation of multilevel/multigrid and domain decomposition methods to explore a variety of real world applications. An important trend in modern numerical simulations is the quick improvement of computer technology that leads to the well known paradigm (see, e.g., [78,179]): high-performance computers make it indispensable to use numerical methods of almost linear complexity in the problem size N, to maintain an adequate scaling between the computing time and improved computer facilities as N increases. In the h-version of the finite element method (FEM), the multigrid iteration realizes an $O(N)$ solver for elliptic differential equations in a domain $\Omega \subset \mathbb{R}^d$ with $N = O(h^{-d})$, where h is the mesh parameter. In the boundary element method (BEM), the traditional panel clustering, fast multi-pole and wavelet based methods as well as the modern hierarchical matrix techniques are known to provide the data-sparse approximations to the arising fully populated stiffness matrices with almost linear cost $O(N_\Gamma \log^q N_\Gamma)$, where $N_\Gamma = O(h^{1-d})$ is the number of degrees of freedom associated with the boundary.

The aim of this book is to introduce a wider audience to the use of a new class of efficient numerical methods of almost linear complexity for solving elliptic partial differential equations (PDEs) based on their reduction to the interface. As in domain decomposition methods, the entire physical domain is decomposed into subregions, and continuity conditions for the corresponding physical fields are formulated on the interfaces between subdomains. For example, it may be the velocity field and fluxes in fluid dynamics or displacements and stresses in structural mechanics. The arising discrete interface equations, called the *Schur complement system*, are derived by the direct FEM approximation to the *elliptic Poincaré-Steklov operators* which provide a mapping between the Dirichlet and Neumann data associated with local subproblems. In this way, the solution to the Schur complement equation recovers a full set of the Cauchy data (traced on the interface), which is important from both physical and computational points of view.

Since our approach essentially deals with interface equations we further call it *interface element method* (IEM). The IEM inherits many beneficial features of both the classical FEM and BEM. In fact, the approximation scheme and the corresponding error analysis are based on the standard Galerkin-FEM. Moreover, the interface equation retains a sparse block structure since every subdomain has an *a priori* fixed number of adjacent subregions. As in BEM, the IEM *reduces a spatial dimension* of the original PDE by one. The Poincaré-Steklov operators defined on subdomain boundaries (pseudodifferential operators) have a nonlocal structure reflecting the integral-differential character of the global interface equation. Nevertheless, and this is the key point, the Schur complement equation can be sparsely approximated and then resolved with complexity $O(N_\Gamma \log^q N_\Gamma)$, which is similar to modern compression schemes in the BEM. Clearly, the IEM has a wider range of applications than the classical BEM because it applies to the case of variable or jumping coefficients as well. On the other hand, in the limit of a large number of subdomains, say of the order of $O(N)$, we arrive at the conventional FEM.

The Poincaré-Steklov operators were first introduced by Steklov [173] and Poincaré [150] and have been further investigated in [4,47,115,117,120,139]. Renewal of interest in the Poincaré-Steklov operators is due to recent advances in the Schur complement matrix compression for the Laplace, biharmonic, Lamé and Stokes equations. Theoretical and numerical results presented in this book are based on the papers [100–102], [115–117], [119]- [125] and [186]- [190] as well as on new materials not previously published. In particular, for equations with piecewise constant coefficients, we describe fast methods of almost linear complexity incorporating data-sparse approximations to the Schur complement and preconditioned iterative solvers applied directly to the interface equation.

No attempt is made to cover the extensive literature on the FEM and BEM in elliptic problems; basic results on the subject can be found in [26,36,45,62,75,140,157] and in [44,76,159,185], respectively. Efficient multilevel/multigrid or domain decomposition methods for solving finite element approximations to elliptic PDEs are well presented, e.g., in [27,37,74,77,151].

In the existing literature, mostly, the preconditioning aspects of the Schur complement method have been addressed; see [152,168,169] and the references therein. Since the Schur complement matrix is in general fully populated, we focus mainly on its accurate and data-sparse approximation with emphasis on 2D applications. However, the scope of Chapters 7.1 – 7.5 is wider and the results can be easily extended to the 3D case. Essential progress in the stiffness matrix compression in BEM was achieved by the panel clustering techniques [76,86], the fast multi-pole methods [68,156] and the wavelet approximations, see, e.g. [18,50,148]. Other interesting ideas have been realized with sparse grids techniques [70,196], Toeplitz/circulant matrices [155,197], and the adaptive *hp*-version of FEM/BEM [166,174]. Modern methods of a data-sparse approximation to a general class of *operator-valued functions* including the inverse of an elliptic operator, are based on the so-called hierarchical matrices introduced in [79], see also [80]- [85] and [17,59,60,67,104,106,158,180] for recent advances. In the case of piecewise analytic coefficients, an $O(N_\Gamma)$

approximation to the elliptic Poincaré-Steklov operators has been recently developed based on the boundary concentrated *hp*-FEM [110, 111].

Although the interface equations in the IEM are well suited for efficient preconditioning in both sequential and parallel implementations, challenging problems arise in the construction of accurate and data-sparse approximations to the discrete Poincaré-Steklov operators that can be represented implicitly by dense Schur complement matrices. Consequently, a direct matrix-vector multiplication process (say, by the multigrid subdomain solvers) has $O(h^{-d})$ complexity. This indicates the significance of a matrix compression to the Schur complement which is systematically addressed in this book. The key difficulty is that the Poincaré-Steklov operators no longer have an explicit representation in terms of boundary integral operators, hence, the panel clustering or wavelet type methods cannot be applied to compress the arising Schur complement matrices. The new technique gets around the problem and comes up with algorithms of linear-logarithmic cost $O(N_\Gamma \log^q N_\Gamma)$. Our approach is based on the regorous error and complexity analysis that includes the analytic theory of elliptic Poincaré-Steklov operators.

As far as the preconditioning issues are concerned, spectrally close approximations (preconditioners) to the Schur complement for the isotropic elliptic operators are well developed [30, 33, 42, 56, 151, 175, 193]. In Chapter 5, we present new robust preconditioners for *anisotropic elliptic operators* with sharply varying or jumping coefficients within a computational domain. This method extends the conventional non-overlapping domain decomposition techniques to the case of highly anisotropic elliptic equations [88, 119], [122]- [125].

We hope that this book will attract the attention of a wide audience to modern numerical methods for solving elliptic PDEs, where the traditional tools from FEM and BEM are involved to complement each other nicely. Applications of the IEM in engineering and scientific computing can be based on the following issues considered in the book:

- asymptotically optimal and well parallelizable algorithms of the complexity $O(N_\Gamma \log^q N_\Gamma)$ to solve elliptic PDEs with piecewise constant (smooth) coefficients based on their interface formulation;
- fast algorithms for solving the Schur complement systems associated with the model Laplace, biharmonic, Lamé and Stokes equations;
- construction of efficient and robust interface preconditioners in the case of highly varying computational parameters (e.g., jumping anisotropic diffusion coefficients, presence of thin/stretched geometries);
- generalisation of the Schur complement method to the case of nested refined meshes, exterior elliptic problems and mortar finite elements.

These notes are mainly addressed to graduate students in the area of numerical analysis, scientific computing, and mathematical physics, to mathematically inclined engineers as well as to experts in numerical analysis.

Here is an outline of the contents. Chapter 1 collects the results on functional analysis in Sobolev spaces and includes the basis of FEM in elliptic problems. In Chapter 2, we survey mapping properties of the elliptic Poincaré-Steklov operators focussing on their relationships with classical boundary integral operators. The finite element error analysis is also presented. Chapters 3 and 4 collect standard results and present new approaches in domain decomposition and multilevel methods applied to the Schur complement system. A robust IEM for highly anisotropic problems is considered in Chapter 5. The basic ideas of the frequency filtering technique and some aspects of its application in the framework of Schur complement method are addressed in Chapter 6. Chapters 7 – 9 form the heart of the book. Chapter 7 is devoted to the data-sparse approximation of the "harmonic" Poincaré-Steklov operators in the case of polygonal domains with further extension to the nested refined meshes and to the exterior boundary value problems. We also consider a matrix compression technique for the biharmonic Poincaré-Steklov operators. Its application to the Lamé equation by using the stress function ansatz is discussed in Chapter 8. Chapter 9 deals with the interface reduction to the Stokes equation based on either the stream function-vorticity formulation (by passing to the biharmonic equation) or on the use of special Poincaré-Steklov operator with respect to the trace of the pressure.

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Gabriel Wittum

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