

Lecture Notes in Mathematics

2015

Editors:

J.-M. Morel, Cachan

B. Teissier, Paris

Subseries:

École d'Été de Probabilités de Saint-Flour

Saint-Flour Probability Summer School



The Saint-Flour volumes are reflections of the courses given at the Saint-Flour Probability Summer School. Founded in 1971, this school is organised every year by the Laboratoire de Mathématiques (CNRS and Université Blaise Pascal, Clermont-Ferrand, France). It is intended for PhD students, teachers and researchers who are interested in probability theory, statistics, and in their applications.

The duration of each school is 13 days (it was 17 days up to 2005), and up to 70 participants can attend it. The aim is to provide, in three high-level courses, a comprehensive study of some fields in probability theory or Statistics. The lecturers are chosen by an international scientific board. The participants themselves also have the opportunity to give short lectures about their research work.

Participants are lodged and work in the same building, a former seminary built in the 18th century in the city of Saint-Flour, at an altitude of 900 m. The pleasant surroundings facilitate scientific discussion and exchange.

The Saint-Flour Probability Summer School is supported by:

- Université Blaise Pascal
- Centre National de la Recherche Scientifique (C.N.R.S.)
- Ministère délégué à l'Enseignement supérieur et à la Recherche

For more information, see back pages of the book and
<http://math.univ-bpclermont.fr/stflour/>

Jean Picard
Summer School Chairman
Laboratoire de Mathématiques
Université Blaise Pascal
63177 Aubière Cedex
France

Franco Flandoli

Random Perturbation of PDEs and Fluid Dynamic Models

École d'Été de Probabilités
de Saint-Flour XL – 2010



Springer

Franco Flandoli
University of Pisa
Department of Applied Mathematics
Via Buonarroti 1
50127 Pisa
Italy
flandoli@dma.unipi.it

ISBN 978-3-642-18230-3 e-ISBN 978-3-642-18231-0
DOI 10.1007/978-3-642-18231-0
Springer Heidelberg Dordrecht London New York

Lecture Notes in Mathematics ISSN print edition: 0075-8434
ISSN electronic edition: 1617-9692

Library of Congress Control Number: 2011923078

Mathematics Subject Classification (2011): 60H15, 60H10, 60J65, 35R60, 35Q35, 35B44, 76BO3

© Springer-Verlag Berlin Heidelberg 2011

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilm or in any other way, and storage in data banks. Duplication of this publication or parts thereof is permitted only under the provisions of the German Copyright Law of September 9, 1965, in its current version, and permission for use must always be obtained from Springer. Violations are liable to prosecution under the German Copyright Law.

The use of general descriptive names, registered names, trademarks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

Cover design: deblik, Berlin

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Preface

Regularization by Noise

The most obvious interpretation of the title *Random Perturbations of PDEs* is *Stochastic Partial Differential Equations* (SPDEs). This is not wrong but the emphasis is that *we start from a PDE* and want to investigate the changes produced by random perturbations. Although it would be of great interest to discuss perturbation of several qualitative properties and objects (like asymptotic behavior, soliton and other special solutions, and so on), we will concentrate only on the fundamental issue of *well posedness*.

The “normal” behavior is that the PDE is well posed and nothing changes passing to the SPDE, except maybe the technics of proofs. In principle it may also happen that the PDE is well posed and the SPDE is not, but this is not common. Much more interesting is, in my opinion, the case when the PDE is *not* well posed but the SPDE *is* well posed. When this happens, we observe what could be called a *regularization by noise*.

Well posedness is not the rule for PDEs arising in fluid dynamics. There are examples of non well posedness and examples where the question is open. Thus regularization by noise in fluid dynamic models would be a very interesting fact, if true. This is the purpose of the research activity reported here. This activity is just at the beginning, since regularization has been proven only for a few simple fluid dynamic models.

As a purpose for a series of Saint Flour lectures, trying to prove that noise restores well posedness of a fluid dynamic equation is certainly a very particular aim. Let me justify this choice by saying that: (a) well posedness of 3-dimensional Navier–Stokes equations is one of the millennium Clay Institute problems (see Fefferman [91]), and: (b) we understand interesting and maybe new features of stochastic analysis and stochastic differential equations (ordinary or partial).

The lack of well posedness mentioned here is of two types. The main one we shall deal with is *lack of uniqueness*. The second one, that we address only very partially, is the *emergence of singularities*. To some extent, the latter phenomenon is even more interesting and intuitive but we have understood it only very partially, until now. Thus it is better to describe some more clear principles behind “uniqueness by noise” and hope the interest in this subject

will drive progresses on the more difficult problem of “interaction between noise and singularities”.

Often, useful series of lectures are devoted to the exposition of general techniques that can be applied to a large variety of problems. We completely lack such a purpose: each example we are able to treat requires its own ideas and techniques. But we hope some of them will lead some researcher to try to find new ones, even if this is far from a mechanical application of mathematical methods.

I would like to thank all those with whom I shared the research work in recent years and whose contribution was essential for the development of the results described in these notes. Finally, I would like to thank my wife Marta, who has more patience than I and she had thought.

Contents

1	Introduction to Uniqueness and Blow-Up	1
1.1	Non Uniqueness in Ordinary Differential Equations	1
1.2	Non uniqueness in Partial Differential Equations	4
1.2.1	Random Perturbations	6
1.3	Definitions of Uniqueness	8
1.3.1	Abstract Scheme	8
1.3.2	Deterministic ODEs	9
1.3.3	Stochastic Differential Equations	10
1.4	Zero-Noise and Selection	13
1.5	Examples of Blow-Up	14
2	Regularization by Additive Noise	17
2.1	Regularization of Functions by Noise: Occupation Measure	17
2.1.1	Examples	17
2.1.2	Is $x \mapsto \mu_{T,x+W(\omega)}$ Continuous in Total Variation?	20
2.1.3	An Estimate for Hölder Functions	21
2.1.4	An Estimate for L^p Functions	25
2.1.5	Summary on Occupation Measure	29
2.2	Regularization of SDE by Additive Noise	31
2.2.1	Main Result	31
2.2.2	Proof of Theorem 2.7	34
2.2.3	Stochastic Flow of Diffeomorphisms	37
2.3	Infinite Dimensional Equations with Additive Noise	40
2.3.1	Introduction	40
2.3.2	Infinite Dimensional Set-Up	42
2.3.3	Uniqueness in Law	46
2.4	Pathwise Uniqueness	53
2.4.1	Finite Dimensional Ornstein–Uhlenbeck and Kolmogorov Equations	56
2.4.2	Proof of Theorem 2.10	62

3 Dyadic Models	71
3.1 Introduction: 3D Euler Equations	71
3.1.1 Fourier Formulation of Euler Equations	73
3.2 The Dyadic Model	74
3.3 Deterministic Results	77
3.3.1 Preliminaries	77
3.3.2 Anomalous Energy Dissipation	78
3.3.3 Numerical Picture of the Anomalous Dissipation	82
3.3.4 Examples of Non-Uniqueness	83
3.3.5 Uniqueness of Positive Solutions	84
3.3.6 Summary and Open Questions	86
3.4 Random Perturbation	87
3.4.1 Preliminary Remarks	87
3.4.2 Definitions and Main Results	89
3.4.3 Auxiliary Linear Equation	92
3.4.4 Girsanov Transform	96
3.4.5 Proof of Theorem 3.4	96
3.4.6 Proof of Theorem 3.5	99
4 Transport Equation	101
4.1 Introduction	101
4.1.1 Linear Transport: Structural Approach	103
4.2 Deterministic Transport Equation	104
4.2.1 Lipschitz Case	104
4.2.2 Weakly Differentiable Case	105
4.3 Stochastic Case	113
4.3.1 Definitions and Preliminaries	113
4.3.2 Renormalized Solutions	115
4.3.3 Main Theorem	117
4.3.4 Representation of the Law of $u(t, x)$	121
4.3.5 Analogy with Stabilization by Noise	122
4.4 Uniqueness by Stochastic Characteristics	123
4.4.1 Stochastic Flow of SDE with Non Regular Drift	123
4.4.2 Proof of Uniqueness Using the Flow: Introduction	123
4.4.3 Distributional Commutator Lemma	125
4.4.4 Convergence of the Commutator Composed with the Flow	127
4.4.5 Final Statement	130
5 Other Models: Uniqueness and Singularities	133
5.1 Negative Examples	133
5.1.1 Linear Transport Equation with Random Coefficients	133
5.1.2 Euler Equation with Too Simple Noise	134
5.1.3 Inviscid Burgers Equation: Non Uniqueness	136
5.1.4 Inviscid Burgers Equation: Blow-Up	137

5.2	Positive Examples	139
5.2.1	Stochastic Continuity Equation	139
5.2.2	Point Vortex Motion	143
5.3	Singularities of Stochastic Schrödinger Equation	149
5.4	Additive Noise in 3D Navier–Stokes Equations	152
5.4.1	About Singularities	153
5.4.2	Markov Selections and Strong Feller Property	154
5.5	Conclusions and Open Problems	155
5.5.1	SPDEs: The Effects of Noise.....	155
5.5.2	What We Have Learned on the Effect of Noise on Uniqueness	156
5.5.3	Other Open Questions	158
References		161