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École d'Été de Probabilités de Saint-Flour

Saint-Flour Probability Summer School

The Saint-Flour volumes are reflections of the courses given at the Saint-Flour Probability Summer School. Founded in 1971, this school is organised every year by the Laboratoire de Mathématiques (CNRS and Université Blaise Pascal, Clermont-Ferrand, France). It is intended for PhD students, teachers and researchers who are interested in probability theory, statistics, and in their applications.

The duration of each school is 13 days (it was 17 days up to 2005), and up to 70 participants can attend it. The aim is to provide, in three high-level courses, a comprehensive study of some fields in probability theory or Statistics. The lecturers are chosen by an international scientific board. The participants themselves also have the opportunity to give short lectures about their research work.

Participants are lodged and work in the same building, a former seminary built in the 18th century in the city of Saint-Flour, at an altitude of 900 m. The pleasant surroundings facilitate scientific discussion and exchange.

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– Université Blaise Pascal
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Random Perturbation of PDEs and Fluid Dynamic Models

École d'Été de Probabilités
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Preface

Regularization by Noise

The most obvious interpretation of the title *Random Perturbations of PDEs* is *Stochastic Partial Differential Equations* (SPDEs). This is not wrong but the emphasis is that *we start from a PDE* and want to investigate the changes produced by random perturbations. Although it would be of great interest to discuss perturbation of several qualitative properties and objects (like asymptotic behavior, soliton and other special solutions, and so on), we will concentrate only on the fundamental issue of *well posedness*.

The "normal" behavior is that the PDE is well posed and nothing changes passing to the SPDE, except maybe the technics of proofs. In principle it may also happen that the PDE is well posed and the SPDE is not, but this is not common. Much more interesting is, in my opinion, the case when the PDE is *not* well posed but the SPDE *is* well posed. When this happens, we observe what could be called a *regularization by noise*.

Well posedness is not the rule for PDEs arising in fluid dynamics. There are examples of non well posedness and examples where the question is open. Thus regularization by noise in fluid dynamic models would be a very interesting fact, if true. This is the purpose of the research activity reported here. This activity is just at the beginning, since regularization has been proven only for a few simple fluid dynamic models.

As a purpose for a series of Saint Flour lectures, trying to prove that noise restores well posedness of a fluid dynamic equation is certainly a very particular aim. Let me justify this choice by saying that: (a) well posedness of 3-dimensional Navier–Stokes equations is one of the millennium Clay Institute problems (see Fefferman [91]), and: (b) we understand interesting and maybe new features of stochastic analysis and stochastic differential equations (ordinary or partial).

The lack of well posedness mentioned here is of two types. The main one we shall deal with is *lack of uniqueness*. The second one, that we address only very partially, is the *emergence of singularities*. To some extent, the latter phenomenon is even more interesting and intuitive but we have understood it only very partially, until now. Thus it is better to describe some more clear principles behind "uniqueness by noise" and hope the interest in this subject

will drive progresses on the more difficult problem of “interaction between noise and singularities”.

Often, useful series of lectures are devoted to the exposition of general techniques that can be applied to a large variety of problems. We completely lack such a purpose: each example we are able to treat requires its own ideas and techniques. But we hope some of them will lead some researcher to try to find new ones, even if this is far from a mechanical application of mathematical methods.

I would like to thank all those with whom I shared the research work in recent years and whose contribution was essential for the development of the results described in these notes. Finally, I would like to thank my wife Marta, who has more patience than I and she had thought.

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