Lecture Notes in Mathematics 2011

Ben Andrews · Christopher Hopper

The Ricci Flow in Riemannian Geometry

A Complete Proof of the Differentiable 1/4-Pinching Sphere Theorem

Springer

Ben Andrews
Australian National University
Mathematical Sciences Institute
ACT 0200 Australia
Ben.Andrews@anu.edu.au

Christopher Hopper
University of Oxford
Mathematical Institute
St Giles' 24-29
OX1 3LB Oxford
United Kingdom
hopper@maths.ox.ac.uk

ISBN: 978-3-642-16285-5 e-ISBN: 978-3-642-16286-2
DOI: 10.1007/978-3-642-16286-2
Springer Heidelberg Dordrecht London New York

Lecture Notes in Mathematics ISSN print edition: 0075-8434
ISSN electronic edition: 1617-9692

Mathematics Subject Classification (2010): 35-XX, 53-XX, 58-XX

Cover design: SPi Publisher Services

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

For in the very torrent, tempest, and as I may say, whirlwind of your passion, you must acquire and beget a temperance that may give it smoothness.

— Shakespeare, *Hamlet.*

Preface

There is a famous theorem by Rauch, Klingenberg and Berger which states that a complete simply connected n-dimensional Riemannian manifold, for which the sectional curvatures are strictly between 1 and 4, is homeomorphic to a n-sphere. It has been a longstanding open conjecture as to whether or not the 'homeomorphism' conclusion could be strengthened to a 'diffeomorphism'.

Since the introduction of the Ricci flow by Hamilton [Ham82b] some two decades ago, there have been several inroads into this problem – particularly in dimensions three and four – which have thrown light upon a possible proof of this result. Only recently has this conjecture (and a considerably stronger generalisation) been proved by Simon Brendle and Richard Schoen. The aim of the present book is to provide a unified expository account of the differentiable 1/4-pinching sphere theorem together with the necessary background material and recent convergence theory for the Ricci flow in n-dimensions. This account should be accessible to anyone familiar with enough differential geometry to feel comfortable with tensors, covariant derivatives, and normal coordinates; and enough analysis to follow standard PDE arguments. The proof we present is self-contained (except for the quoted Cheeger–Gromov compactness theorem for Riemannian metrics), and incorporates several improvements on what is currently available in the literature.

Broadly speaking, the structure of this book falls into three main topics. The first centres around the introduction and analysis the Ricci flow as a geometric heat-type partial differential equation. The second concerns Perel'man's monotonicity formulæ and the 'blow up' analysis of singularities associated with the Ricci flow. The final topic focuses on the recent contributions made – particularly by Böhm and Wilking [BW08], and by Brendle and Schoen [BS09a] – in developing the necessary convergence theory for the Ricci flow in n-dimensions. These topics are developed over several chapters, the final of which aims to prove the differentiable version of the sphere theorem.

The book begins with an introduction chapter which motivates the pinching problem. A survey of the sphere theorem's long historical development is discussed as well as possible future applications of the Ricci flow.

As with any discussion in differential geometry, there is always a labyrinth of machinery needed before any non-trivial analysis can take place. We present some of the standard and non-standard aspects of this in Chap. 1. The chapter's focus is to set the notational conventions used throughout, as well as provide supplementary material needed for future computations – particular for those in Chaps. 2 and 3. Careful attention is paid to the construction of the connection and curvature on various bundles together with some non-standard aspects of the pullback bundle structure. We refer the reader to [Lee02, Lee97, Pet06, dC92, Jos08] as additional references with respect to this background material.

In Chap. 2 we look at some classical results related to Harmonic map heat-flow between Riemannian manifolds. The inclusion of this chapter serves as a gently introduction to the techniques of geometric analysis as well as provides good motivation for the Ricci flow. Within, we present the convergence result of Eells and Sampson [ES64] with improvements made by Hartman [Har67].

After establishing this, Chap. 3 introduces the Ricci flow as a geometric parabolic equation. Some basic properties of the flow are discussed followed by a detailed derivation of the associated evolution equations for the curvature tensor and its various traces. Thereafter we give a brief survey of the sphere theorem of Huisken [Hui85], Nishikawa [Nis86] and Margerin [Mar86] together with the algebraic decomposition of the curvature tensor.

Short-time existence for the Ricci flow is discussed in Chap. 4. We follow the approach first outlined by DeTurck [DeT83] which relates Ricci flow to Ricci–DeTurck flow via a Lie derivative. A discussion on the ellipticity failure of the Ricci tensor due to the diffeomorphism invariance of the curvature is also included.

Chapter 5 discusses the so-called Uhlenbeck trick, which simplifies the evolution equation of the curvature so that it can be written as a reaction-diffusion type equation. This will motivate the development of the vector bundle maximum principle of the next chapter. We present the original method first discussed in [Ham86], and improved in [Ham93], which uses an abstract bundle and constructs an identification with the tangent bundle at each time. Thereafter we introduce a new method that looks to place a natural connection on a 'spatial' vector bundle over the space-time manifold $M \times \mathbb{R}$. We will build upon this space-time construction in subsequent chapters.

In Chap. 6 we discuss the maximum principle for parabolic PDE as a very powerful tool central to our understanding of the Ricci flow. A new general vector bundle version, for heat-type PDE of section $u \in \Gamma(E \to M \times \mathbb{R})$ over the space-time manifold, is discussed here. The stated vector bundle maximum principle, Theorem 7.15 and the related Corollary 7.17, will provide the main tools for the convergence theory of the Ricci flow discussed in later chapters. Emphasis is placed on the 'vector field points into the set' condition as it correctly generalises the null-eigenvector condition of Hamilton [Ham82b]. The related convex analysis necessary for the vector bundle maximum principle is discussed in Appendix B – where we use the same conventions as that

of the classic text [Roc70]. The maximum principle for symmetric 2-tensors is also discussed as well as applications of the Ricci flow for 3-manifolds.

The parabolic nature of the Ricci flow is further developed in Chap. 7 where regularity and long-time existence is discussed. We see that the Ricci flow enjoys excellent regularity properties by deriving global Shi estimates [Shi89]. They are used to prove long-time existence soon thereafter.

Chapter 8 look at a compactness theorem for sequences of solutions to the Ricci flow. The result originates in the convergence theory developed by Cheeger and Gromov. We use the regularity of previous chapter to give a proof of the compactness theorem for the Ricci flow, given the compactness theorem for metrics; however, we will not give a proof of the general Cheeger–Gromov compactness result. It has natural applications in the analysis of singularities of the Ricci flow by 'blow-up' – which we will employ in the proof of the differentiable sphere theorem.

Chapters 9 and 10 aim to establish Perel'man's local noncollapsing result for the Ricci flow [Per02]. This will provide a positive lower bound on the injectivity radius for the Ricci flow under blow-up analysis. We also discuss the gradient flow formalism of the Ricci flow and Perel'man's motivation from physics [OSW06, Car10].

The work of Böhm and Wilking [BW08], in which whole families of preserved convex sets for the Ricci flow are derived from an initial one, is presented in Chaps. 11 and 12. Using this we will be able to argue, in conjunction with the vector bundle maximum principle, that solutions of the Ricci flow which have their curvature in a given initial cone will evolve to have constant curvature as they approach their limiting time. Chapter 11 focuses on the algebraic decomposition of the curvature and the required family of scaled transforms. In particular we use the inherent Lie algebra structure (discussed in Appendix C) related to the curvature to elucidate the nature of the reaction term in the evolution equation for the curvature. The key result (Theorem 12.33) is that these transforms induce a change in the reaction terms which does not depend on the Weyl curvature, and so can be computed entirely from the Ricci tensor and expressed in terms of its eigenvalues. Chapter 12 uses these explicit eigenvalues to generate a one-parameter family of preserved convex cones that are parameterised piecewise into two parts; one to accommodate the initial behaviour of the cone, the other to accommodate the required limiting behaviour. Thereafter we discuss the formulation of generalised pinching sets. The main result of this section, Theorem 13.8, provides the existence of a pinching set simply from the existence of a suitable family of cones.

In Chap. 13 we discuss the positive curvature condition on totally isotropic 2-planes, first introduced by Micallef and Moore [MM88], as a possible initial convex cone. We show, using ideas from [BS09a, Ngu08, Ngu10, AN07], that the positive isotropic curvature (PIC) condition is preserved by the Ricci flow; as is the positive complex sectional curvature (PCSC) condition. We also give a simplified proof that PIC is preserved by the Ricci flow by working directly

with the complexification of the tangent bundle. In order to relate the PIC condition to the 1/4-pinching sphere theorem, we present the argument of Brendle and Schoen [BS09a] that relates the 1/4-pinching condition with the PIC condition on $M \times \mathbb{R}^2$. The result, i.e. Corollary 14.13, shows that M is a compact manifold with pointwise 1/4-pinched sectional curvature, then $M \times \mathbb{R}^2$ has positive isotropic curvature.

Chapter 14 brings the discussion to a climax. Here we finally give a proof of the differentiable 1/4-pinching sphere theorem from the material presented in earlier chapters. In the final section we outline a general convergence result due to Brendle [Bre08] which looks at the weaker condition of PIC on $M \times \mathbb{R}$.

A synopsis of the chapter progressions and inter-relationships is summarised by the following diagram:

$$\begin{tikzcd}
 & & & & & 4 \arrow[r] & 7 \arrow[d] \\
 & & & & 9 \arrow[r] & 10 \arrow[r] & 8 \arrow[d] \\
3 \arrow[r] & 5 \arrow[r] & 6 \arrow[r] & 11 \arrow[r] & 12 \arrow[r] & 13 \arrow[r] & 14 \\
 & & B \arrow[u] & C \arrow[u] & & &
\end{tikzcd}$$

Here the main argument is represented along the horizontal together with the supplementary appendices. The regularity, existence theory and blow-up analysis are shown above this.

This book grew from my honours thesis completed in 2008 at the Australian National University. I would like to express my deepest gratitudes to my supervisor, Dr Ben Andrews, for without his supervision, assistance and immeasurable input this book would not be possible. Finally, I would like to thank my parents for their continuous support and encouragement over the years.

July, 2009 *Christopher Hopper*

Contents

Notation and List of Symbols

Γ_{ij}^k	*Christoffel symbol* of a connection ∇ w.r.t. a local frame (∂_i).
(x^i)	A *local coordinate chart* for a neighbourhood U with a local chart $x = (x^i) : U \to \mathbb{R}^n$.
Curv	*Algebraic curvature operators.*
I, id	The identities on $S^2(\bigwedge^2 V^*)$ and $S^2(V)$. N.B. $I = \mathrm{id} \wedge \mathrm{id}$.
inj	*Injectivity radius.*
$\Delta_{g,h}$	*Harmonic map Laplacian* w.r.t. the domain metric g and codomain metric h.
$\bigwedge^2 V$	*Second exterior power* of the vector space V.
$\mathcal{L}_X$	*Lie derivative* w.r.t. the vector field X.
$\mathfrak{Met}$	*Space of metrics.*
$\mathcal{N}_x A$	*Normal cone* to A at x.
ν	*Unit outward normal.*
$\owedge$	*Kulkarni-Nomizu product.*
${}^f\nabla$	*Pullback connection* on f^*E.
f^*E	*Pullback bundle* of E by f.
$\xi_f, (\xi_i)_f$	*Restriction* of $\xi, \xi_i \in \Gamma(E)$ to f.
R, $R_{ijk\ell}$	*Riemannian curvature tensor.*
Ric, R_{ij}	*Ricci curvature tensor.*
Scal	*Scalar curvature tensor.*
$\Gamma(E)$	The space of *smooth sections* of a vector bundle $\pi : E \to M$.
$\mathfrak{S}$	*Spatial tangent bundle.*
$\mathrm{Sym}^2 T^*M$	*Symmetric* $(2,0)$*-bundle* over M.
$S^2(U)$	*Symmetric tensor space* of U.
$T_\ell^k(V)$	The set of all multilinear maps $(V^*)^\ell \times V^k \to \mathbb{R}$ over V.
$T_\ell^k M$	(k,ℓ)*-tensor bundle* over a manifold M.
$\mathscr{T}_\ell^k(M)$	The space of (k,ℓ)*-tensor fields* over M, i.e. $\Gamma(\otimes^k T^*M \otimes^\ell TM)$.
$\mathcal{T}_x A$	*Tangent cone* to A at x.
$d\mu$, $d\mu(g)$	*Volume form* with respect to a metric g.
$d\sigma$	*Volume form on a hypersurface or boundary* of a manifold.
$\mathscr{X}(M)$	The space of *vector fields*, i.e. $\Gamma(TM)$.