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Topics in Algebraic and Topological K-Theory

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It shows a snowman built by the winter school participants.

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Foreword

The five articles of this volume evolved from the lecture notes of Swisk¹, the Sedano Winter School on K -theory held in Sedano, Spain, during the week January 22–27 of 2007. Lectures were delivered by Paul F. Baum, Carlo Mazza, Ralf Meyer, Marco Schlichting, Bertrand Toën and myself, for a public of 45 participants. The school was supported by the Ministerio de Educación y Ciencia and the Proyecto Consolider Mathematica of Spain. Funding to cover expenses of US based participants was provided by NSF, through a grant to C.A. Weibel, who was responsible, first of all, for preparing a successful funding application, and then for managing the funds. The local committee, composed of N. Abad and E. Ellis, were in charge of conference logistics. Marco Schlichting collaborated in the Scientific Committee. As organizer of the school and editor of this volume, I am indebted to all these people and institutions for their support, and to my fellow coauthors for their contributions.

Guillermo Cortiñas

¹ The webpage of the school can be found at <http://cms.dm.uba.ar/Members/gcorti/workgroup.swisk/index.html.html>

Preface

This book evolved from the lecture notes of Swisk, the Sedano Winter School on K -theory held in Sedano, Spain, during the week January 22–27 of 2007. It intends to be an introduction to K -theory, both algebraic and topological, with emphasis on their interconnections. While a wide range of topics is covered, an effort has been made to keep the exposition as elementary and self-contained as possible.

Since its beginning in the celebrated work of Grothendieck on the Riemann–Roch theorem, applications of K -theory have been found in a variety of subjects, including algebraic geometry, number theory, algebraic and geometric topology, representation theory and geometric and functional analysis. Because of this, mathematicians from each of these areas have become interested in the subject, and they all look at it from their own perspective. On the one hand, this is the richness and appeal of K -theory. On the other hand, it makes it hard to see a global perspective. For example it is not often that an algebraic K -theorist, coming, say, from the algebraic geometry side of the subject, and a topological K -theorist, coming from the functional analysis side, meet together in the same K -theory conference. Thus it is not uncommon to find that algebraic and topological K -theory are regarded as distinct subjects altogether. These notes modestly attempt to illustrate current developments in both branches of the subject, and to emphasize their contacts.

The book is divided into five articles.

The first two are concerned with Kasparov’s bivariant K -theory of C^* -algebras and its role in the Baum–Connes conjecture. If G is a locally compact group and A and B are two separable C^* -algebras equipped with a G -action, the Kasparov bivariant K -theory group $KK^G(A, B)$ is defined as the homotopy classes of G -equivariant Hilbert (A, B) bimodules equipped with a suitable Fredholm operator. Kasparov defines an associative product

$$K^G(A, B) \otimes K^G(B, C) \rightarrow K^G(A, C)$$

There is an additive category KK^G whose objects are the separable G - C^* -algebras, so that $KK^G(A, B) = \text{hom}_{KK^G}(A, B)$ and composition is given by the Kasparov product. This category is related to usual category G - C^* -Alg of G - C^* -algebras and

equivariant $*$ -homomorphism by means of a functor $\iota : G\text{-}C^*\text{-Alg} \rightarrow KK^G$. The functor ι has the following properties:

- (Stability) If $A \in G\text{-}C^*\text{-Alg}$, and H_1, H_2 are nonzero G -Hilbert spaces, then $\iota(A \otimes K(H_1) \rightarrow A \otimes K(H_1 \oplus H_2))$ is an isomorphism.
- (Split Exactness) If $A \xrightarrow{j} B \xrightarrow{p} C$ is a short exact sequence of $G\text{-}C^*$ -algebras, split by a G -equivariant homomorphism $s : C \rightarrow B$, then $(\iota(j), \iota(s)) : \iota(A) \oplus \iota(C) \rightarrow \iota(B)$ is an isomorphism.

Moreover ι is universal (initial) among stable, split exact functors to additive categories. Kasparov theory has many other important properties. To mention one, consider the case when $G = \{1\}$ is the trivial group, and we take \mathbb{C} as the first variable; then

$$K_0(B) = KK(\mathbb{C}, B)$$

is the usual Grothendieck group. Because topological K_1 of a C^* -algebra B is just K_0 of the suspension of B , we also have

$$K_1^{\text{top}}(B) = KK(\mathbb{C}, SB)$$

Thus the whole topological K -theory is recovered from KK , since K^{top} is 2-periodic.

Another application of equivariant KK is in the definition of equivariant K -homology, which plays a fundamental role in the Baum–Connes conjecture. A Hausdorff, locally compact, second countable space X equipped with an action of G by homeomorphisms is called *proper* if the map

$$G \times X \rightarrow X \times X, \quad (g, x) \mapsto (gx, x)$$

is proper, that is, if the inverse image of any compact subspace is compact. A G -subspace $\Delta \subset X$ is called G -compact if it is proper and the quotient $G \backslash \Delta$ is compact. The equivariant K -homology of a proper G -space X is

$$K_*^G(X) = \text{colim}_{\Delta \subset X} KK_*^G(C_0(\Delta), \mathbb{C})$$

Here the colimit is taken over all G -compact subspaces $\Delta \subset X$; C_0 is the C^* -algebra of continuous functions vanishing at infinity, and $KK_*^G(A, \mathbb{C}) = KK^G(SA, \mathbb{C})$.

The Baum–Connes conjecture proposes a description of the topological K -theory of the reduced C^* -algebra $C_r^*(G)$ of a locally compact, Hausdorff, second countable group G in terms of G -equivariant K -homology of the universal (final) proper G -space. There is a map

$$K_*^G(\underline{E}G) \rightarrow K_*^{\text{top}}(C_r^*(G))$$

called the assembly map, and the conjecture says it is an isomorphism. The proper G -space $\underline{E}G$ is characterized up to homotopy by the property that any proper G -space X maps to $\underline{E}G$ and that any two such maps are homotopic. The reduced C^* -algebra is defined as follows. If G is a locally compact, second countable group, and μ is a left invariant Haar measure on G , then one can form the separable Hilbert space $H = L^2(G, \mu)$ of square-integrable functions on G . The algebra $C_c(G)$

of compactly supported continuous functions $G \rightarrow \mathbb{C}$ with convolution product is faithfully represented inside the algebra $\mathcal{B}(H)$ of bounded operators on H , and $C_r(G)$ is the norm completion of $C_c(G)$.

The conjecture is known to be true for wide classes of groups; no counterexamples are known. There is also a more general version of the conjecture relating the equivariant K -homology of $\underline{E}G$ with coefficients in a separable G - C^* -algebra A with the topological K -theory of the reduced C^* -algebra of G with coefficients in A , $C_r(G, A)$. The latter conjecture is also known for large classes of groups, and is expected to be true in many cases.

The Baum–Connes conjecture is related to a great number of conjectures in functional analysis, algebra, geometry and topology. Most of these conjectures follow from either the injectivity or the surjectivity of the assembly map. A significant example is the Novikov conjecture on the homotopy invariance of higher signatures of closed, connected, oriented, smooth manifolds. This conjecture follows from the injectivity of the rationalized assembly map.

The first article of this volume, K -theory for group algebras, written by P. Baum and R. Sánchez-García, introduces the subject step by step, beginning with the definition of a C^* -algebra, passing through K -theory of C^* -algebras and its connection with Atiyah–Hirzebruch theory, to the general formulation of the Baum–Connes conjecture with coefficients, and of Kasparov’s equivariant KK -theory. The latter is introduced in terms of homotopy classes of Hilbert bimodules.

Universal Coefficient Theorems and assembly maps in KK -theory, by R. Meyer, looks at KK -theory and the Baum–Connes conjecture from the point of view of triangulated categories. Equivariant Kasparov theory is introduced using its universal property, and it is explained how this category can be triangulated. The Baum–Connes assembly map is constructed by localising the Kasparov category at a suitable subcategory. Then a general machinery to construct derived functors and spectral sequences in triangulated categories is explained. This produces various generalizations of the Rosenberg–Schochet Universal Coefficient Theorem.

The next article, Algebraic versus topological K -theory: a friendly match, by G. Cortiñas, attempts to be a bridge between the algebraic and topological branches. It presents various variants of algebraic K -theory of rings, including Quillen’s, Karoubi–Villamayor’s, and Weibel’s homotopy algebraic K -theory, denoted respectively K , KV and KH . These variants of algebraic K -theory differ in their behavior with respect to homotopy and excision. Both KV and KH are invariant under polynomial homotopy; if A is any ring, we have $KV_*(A[t]) = KV_*(A)$ and similarly for KH . On the other hand the identity $K_*(A) = K_*(A[t])$ holds in particular cases (e.g. when A is noetherian regular) but not in general. As to excision, if

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

is an exact sequence of (nonunital) rings, then there is a long exact sequence ($n \in \mathbb{Z}$)

$$KH_{n+1}(C) \longrightarrow KH_n(A) \longrightarrow KH_n(B) \longrightarrow KH_n(C)$$

A similar sequence holds for KV under the additional assumption that the sequence be split by a ring homomorphism $C \rightarrow B$. The sequence

$$K_{n+1}(C) \rightarrow K_n(A) \rightarrow K_n(B) \rightarrow K_n(C)$$

is exact for $n \leq 0$, but not for $n \geq 1$, in general. Topological K -theory of topological algebras also has several variants, essentially depending on the type of algebras considered. The topological K -theory of Banach algebras is invariant under continuous homotopies; that for locally convex algebras is invariant under C^∞ -homotopies. Both satisfy excision and (when suitably stabilized) Bott periodicity: $K_n^{\text{top}} = K_{n+2}^{\text{top}}$.

If A is a topological algebra, there is a comparison map

$$K_*(A) \rightarrow K_*^{\text{top}}(A)$$

which is not an isomorphism in general.

Cortiñas' article emphasizes the connections –both formal and concrete– between the algebraic and topological counterparts. For example, Bott periodicity for topological K -theory and the fundamental theorem in algebraic K -theory (which computes the K -groups of the Laurent polynomials) are introduced in a way that makes it clear that each of them is the counterpart of the other. As a concrete connection between algebraic and topological K -theory, the question of whether the comparison map $K_*(A) \rightarrow K_*^{\text{top}}(A)$ between the algebraic and topological K -theory of a given topological algebra A is an isomorphism is discussed; Karoubi's conjecture (Suslin–Wodzicki's theorem) establishes that the answer is affirmative for stable C^* -algebras. Proofs of this theorem and of some of its variants are given.

The last two articles approach algebraic K -theory from a categorical point of view.

Higher algebraic K -theory (after Quillen, Thomason and others), by M. Schlichting, introduces higher algebraic K -theory of schemes; emphasis is on the modern point of view where structure theorems on derived categories of sheaves are used to compute higher algebraic K -groups. There are many results in the literature about the structure of triangulated categories, and virtually all of them translate into results about higher algebraic K -groups. The link is provided by an abstract localization theorem due to Thomason and Waldhausen, which –omitting hypothesis– says that a short exact sequence of triangulated categories gives rise to a long exact sequence of algebraic K -groups. This theorem, and its applications, are the heart of the article. Among the main applications presented in the article is Thomason's Mayer–Vietoris theorem, which says that if X is a quasi-compact, quasi-separated scheme and U and V are open quasi-compact subschemes, then there is a long exact sequence

$$K_{n+1}(U \cap V) \rightarrow K_n(X) \rightarrow K_n(U) \oplus K_n(V) \rightarrow K_n(U \cup V) \rightarrow K_{n-1}(X)$$

Although the particular case of this result for regular noetherian separated schemes follows from Quillen's early work in the 1970s, the full generality was obtained only

twenty years later, by Thomason. The use of derived categories is essential in its proof. Another application is Thomason's blow-up formula. If $Y \subset X$ is a regular embedding of pure codimension d with X quasi-compact and separated, and X' is the blow-up of X along Y , then

$$K_*(X') = K_*(X) \oplus \bigoplus_{i=1}^{d-1} K_*(Y)$$

The methods explained in Schlichting's paper can also be applied to any of the other (co-) homology theories which satisfy an analog of Thomason–Waldhausen's localization theorem; these include Hochschild homology, (negative, periodic, ordinary) cyclic homology, topological Hochschild (and cyclic) homology, triangular Witt groups and higher Grothendieck–Witt groups (the last two when 2 is invertible).

Lectures on dg-categories, by B. Toën, provides an introduction to this theory, which is deeply intertwined with K -theory. The connection comes from the fact that the categories of complexes of sheaves on a scheme are dg-categories. The approach to the subject emphasizes the localization problem, in the sense of category theory. In the same way that the notion of complexes is introduced for the need of derived functors, dg-categories are introduced here for the need of a “derived version” of the localization construction. The existence and properties of this localization are then studied. The notion of triangulated dg-categories, which is a refined version of the usual notion of triangulated categories, is presented, and it is shown that many invariants (such as K -theory, Hochschild homology, . . .) are invariants of dg-categories, though it is known that they are not invariants of triangulated categories. Finally the notion of saturated dg-categories is given and it is explained how they can be used in order to define a secondary K -theory.

June, 2010

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