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Geometric Theory of Discrete Nonautonomous Dynamical Systems

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Dedicated to my teacher Bernd Aulbach (1947–2005)

Preface

Πάντα χωρεῖ καὶ ουδὲν μένει
Heracleitus, 502 BC

The monograph in your hands deals with difference equations, or in a terminology equivalent for us, with recursions, iterations and discrete dynamical systems. Such iterative procedures are omnipresent in mathematics, as well as in its related sciences – for approximation as well as for modelling purposes. Their history can be traced back to Pythagoras (triangular numbers, $\sim$500 BC), Euclid (continued fractions, $\sim$250 BC) and Archimedes (computation of π, $\sim$200 BC), that is, the beginning of mathematics as we know it today. Early systematic approaches to difference equations as independent mathematical discipline appeared in the 1920–1950s in form of classical monographs, like for instance [42, 166, 179, 303]. These early contributions are basically concerned with a linear theory and connections to the field of functional equations. After that, corresponding research stagnated somehow and difference equations found themselves in the shadow of their continuous counterpart, namely evolutionary differential equations of various kind. However, differing from classical results obtained in the 1950s and before, in recent decades nonlinear problems and phenomena reentered the center of interest and finally led to an extensive theory of discrete dynamical systems. One reason for their popularity is definitely that already very simple equations show a surprisingly complex dynamical behavior, like, e.g., the tent-map, the logistic equation or Smale's horseshoe map. Fields like "chaos theory" draw a strong motivation from such examples which additionally serve as prototypes to understand more complex phenomena. Indeed, over the past 20 years the mathematical community observed a renaissance of difference equations. Several new journals have been successfully introduced,[1] conference series are established and various new monographs appeared (e.g., [3,4,84,103,133,175,248,276,281,289,294,297,334,425]). In the course of this revival also somewhat philosophical arguments to support discrete dynamics have occurred. Actually, many laws of nature are intrinsically

[1] Discrete and Continuous Dynamical Systems, Journal of Difference Equations and Applications, Advances in Difference Equations, International Journal of Difference Equations, etc.

discrete (cf. [132, 225, 302, 466]), providing the insight that a "correct" description of our world might be a discrete one. As a conclusion one can state that difference equations form a theory of its own right and are worth to investigate.

Continuous to discrete: There is admittedly a strong analogy between the theories of discrete and of continuous dynamical systems, which even led to a unifying calculus (cf. [204]). Yet, particularly in low dimensions discrete models tend to have a more complex behavior due to the fact of nonexistent backward solutions, or missing topological constraints like connected solution curves. For that reason alone, it is unjustified that the continuous theory is usually preferred when it comes to a rigorous presentation in the literature, while its discrete counterpart is labeled as "analogous" or proofs are attributed to work "along the lines".

As a matter of course, a key application for difference equations and discrete dynamical systems comes from various discretizations of (evolutionary) differential equations. Here, "discretization" can have different meanings and discrete approaches are quite beneficial for the dynamical systems theory as a whole:

- For various problems it is convenient to study the (discrete) time-h-map $\varphi(h)$, instead of a (continuous) semiflow $(\varphi(t))_{t\geq 0}$ itself – for example in topological linearizations (cf. [200]) and to construct invariant manifolds (cf. [83, 285, 343], or [169] dealing with invariant manifolds for PDEs on unbounded domains). Another source for such applications are abstract functional differential equations; here, in certain cases no variation of constants formula for the continuous problem is known and one has to work with the corresponding time-h-map of the generated semiflow to obtain invariant manifolds for the continuous flow (cf. [285, Sect. 4]).
- Poincaré (or return) maps are a popular tool to study the behavior of periodic continuous motions, in particular since they offer a possibility to reduce the dimension of a problem by 1 (cf., for instance, [9, p. 320ff], [227, pp. 17–25] or [319, pp. 56–62]).
- The asymptotic behavior of abstract nonautonomous (linear) evolutionary equations is often studied using difference equations, where the continuous evolution operator is restricted to the integers. Using the resulting discrete equation, it is more convenient to deduce results on the long term behavior, and then to show that they extend to the continuous problem (see [421] for stability results, [88, 201, 299, 300, 369, 372] for exponential dichotomies or [301] for a Fredholm theory).
- Last but not least, numerical schemes applied to differential equations canonically lead to difference equations and it is important to have a sufficiently rich discrete theory at our disposal (cf. [447]).

In conclusion, even within the field of dynamical systems it seems legitimate to claim that the continuous theory benefits more from the discrete one than the other way around. As a consequence, discrete dynamical systems and difference equation require an adequate presentation.

Even beyond that, from a modeling and simulation perspective, it is frequently more reasonable and sometimes plainly honest to work with discrete models right from the beginning, instead of enforcing a continuous model and then to discretize it in order to make it solvable on a computer.

Autonomous to nonautonomous: Beyond our above considerations, the recent years have seen a growing interest in nonautonomous problems, i.e., equations whose right-hand sides explicitly depend on time or chance (see, e.g., the upcoming monographs [79, 266]). Indeed, nonautonomous equations allow more realistic models, since they enable us to include seasonal influences, as well as regulation, controlling, modulating or random effects. In concrete situations this is realized in a way that constant parameters are replaced by time-dependent sequences (*parametric perturbations*) or driven by external (decoupled) equations (*driven equations*). Moreover, in contrast to an already stochastic approach, the advantage of deterministic nonautonomous models is that their results are easier to interpret (cf. [454]) and to tackle, because they require only point estimation of constants instead to specify complete distributions for random variables as in the case of stochastic models. Further reasons illustrating the importance of a nonautonomous deterministic theory are as follows:

- It canonically appears in a seemingly autonomous setting, like, e.g., to study the behavior near nonconstant reference solutions or in the construction of invariant foliations (see, for example, [33, 83, 89, 157]). So why not considering nonautonomous equations right from the beginning?
- Time-adaptive discretization schemes lead to nonautonomous problems (cf., e.g., [55, 173, 267, 268]). In fact, so far analytical discretization theory essentially never leaves the framework of autonomous dynamical systems. Thus, often schemes with constant stepsizes are considered, which from an applied point of view and referring to adaptive schemes is a rather artificial point of view.
- Results from the deterministic theory of difference equations are applicable to random difference equations on a path-wise basis (cf. [12, pp. 50, Sect. 2.1] or [459]), i.e., by considering concrete realizations of random variables.

Our approach to nonautonomous dynamical systems is based on 2-parameter semigroups (or discrete processes) rather than on *skew product dynamics* – a notion coined in a series of papers by Sacker and Sell (see, e.g., [415–417, 419] or the memoirs [418]) during the 1970s. In a skew product framework, one enlarges the state space by encoding the time-dependence using a flow on the so-called base space (cf. [429]). Hence, one is in the convenient position to apply methods from classical autonomous dynamical systems. Skew product dynamics is motivated by re-capturing the geometric flavor that is inherent to autonomous dynamics and also various hierarchical or triangular systems fall into the abstract skew product category. Nevertheless, in contrast to the admittedly elegant skew product setting, we avoid the resulting topologically subtle questions and assumptions on the particular time dependencies, which guarantee that the corresponding base space becomes compact.

Geometric theory to discretizations: A central motivation for this work is to bring together ideas and results from three related, yet different areas of applied mathematics mentioned above: Difference Equations, (Nonautonomous) Dynamical Systems and (Theoretical) Numerical Analysis. They are obviously related in the sense that iterations are of central importance. But on the other hand, unfortunately they rarely rely on each other, the corresponding scientific communities hardly overlap and chances for synergetic effects are missed. We intend to introduce some modern concepts from the recent theory of nonautonomous dynamical systems into the seemingly classical field of difference equations. Within this broad field, we restrict on certain aspects of what is commonly known as "qualitative" or more precisely as "geometric theory".

This area was essentially initiated by Poincaré and Lyapunov over a century ago. It aims to identify certain invariant subsets of the state space, which "prescribe" the long-term behavior of a system. First, it deals with questions of the existence of special solutions (equilibria, periodic, almost-periodic or complete bounded solutions, etc.) or collections of solutions (*invariant manifolds*) with a particular growth behavior, as well as their stability and domain of attraction. Second, it intends to identify prototype system which are particularly simple but share the essential dynamics (topological conjugation and *structural stability*). Third, also addressed are related global questions, like starting from an "arbitrary" initial value, what can be said about the long-term dynamics (or the *(global) attractor*). For a broader overview, we refer to, for instance, [12, 192, 198, 201, 211, 245, 253, 348, 432, 434, 462].

To a minor extent, we are interested in discretization theory or what is nowadays known as *numerical dynamics*. The essential goals in this field are (1) to investigate and determine features of continuous dynamical systems which persist under discretization, and (2) to obtain convergence results for small stepsizes or spatial discretization meshes. For a survey, see [54, 172, 193, 222, 313, 445–447].

This monograph aims to extend the above complex of questions and to provide a consistent reference. In doing so, we throughout deal with *nonautonomous* discrete equations. In order to possess stability properties required in discretization theory, they are allowed to be *implicit*. Furthermore, their state spaces can be *infinite-dimensional* and time-dependent. This set-up allows immediate *applications* to various temporal and full discretizations of evolutionary differential equations, i.e., to address the aspect (1) above. However, we clearly point out to focus on the persistence aspect of numerical dynamics and totally neglect the crucial convergence questions addressed in aspect (2). Yet, we hope to lay down the basics for future applications towards convergence issues.

At hand is particularly a rather complete approach to invariant manifold theory for implicit nonautonomous difference equations in Banach spaces. Here, differing from various approaches in the literature, fully implicit numerical schemes fit into our set-up. In detail, our contents can be summarized as follows:

- The first chapter introduces 2-parameter semigroups acting on the extended state space – our notion to describe nonautonomous dynamics. We consistently use the concept of *pullback convergence*. Accordingly, the corresponding limit sets and attractors are sequences of sets rather than single sets as in the classical

autonomous situation. Under various compactness assumptions, we provide criteria for their existence and derive basic properties. Moreover, we illustrate how these objects simplify to known-ones for the periodic or autonomous case.

- A quite flexible notion for difference equations is discussed in Chap. 2, which includes implicit discretization methods. We investigate conditions for them to generate 2-parameter semigroups, to be dissipative or to possess (global) attractors; in doing so, we particularly address one-step methods. Surely, the nonautonomous stability theory is in part classical, but understandably more complex than in the autonomous (or periodic) situation. Yet, we present and relate it to attraction and stability notions based on pullback convergence. Finally, simplifications in the periodic and autonomous case are illuminated.
- The theory of linear difference equations in Chap. 3 serves as foundation for our following perturbation arguments. Here, stability is a property of the whole system and not only of single solutions. After that we briefly touch periodic equations and Floquet theory. Exponential dichotomies and more general splittings turn out to be an appropriate hyperbolicity notion in our nonautonomous setting. In addition, we provide several results discussing the behavior of splittings under perturbation.
- Our time-dependent counterpart to classical invariant manifolds are so-called invariant fiber bundles. We provide an abstract approach, which as application, yields bundles associated to given reference solutions (local theory), as well as a discrete version of inertial manifolds (global theory). In doing so, we prove results on invariant foliations and asymptotic phases. Smoothness issues are tackled as well, using an elementary approach which is essentially based on the contraction mapping principle. This allows us to obtain Taylor approximations of local invariant fiber bundles. We also describe a numerical scheme to compute global approximations.
- Finally, our achievements from the previous chapter, allow to deduce results on topological decoupling and linearization. They include a generalized Hartman–Grobman theorem for invertible nonautonomous problems with nonhyperbolic spectrum. We can get rid of the invertibility assumption when shifting to the concept of solution conjugacy. The latter is still sufficient to deduce smoothness properties of invariant fiber bundles.

Every chapter is supplemented by an illustrating section dealing with applications. It extends our so far theoretical approach and illustrates that the previous results and methods are applicable to discretizations of various evolutionary differential equations, like for example of functional differential-, reaction-diffusion- or abstract type. Moreover, a final concluding section points out the relevant literature, provides historical context and indicates directions for further research.

The appendix collects a number of helpful results needed in the text. It addresses discrete inequalities, various fixed point and global inversion theorems, as well as explanations on smooth functions. In particular, we provide a survey on smooth norms, which are important to construct global extensions of differentiable mappings and locally invariant fiber bundles.

The monograph is linearly written with the exception of some references to the appendix and that certain applications in Sect. 2.6 require a lookahead to independent results from Sect. 3.7. As a general philosophy behind these notes, it is our intention to provide explicit estimates and constants to a large extent. This might lead to a technical appearance, but enables us to obtain quantitative results on, e.g., growth rates of solutions, the radius of absorbing sets or the dimension of (attractive) invariant manifolds. Understandably, the references have bias on discrete dynamics.

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Munich, *Christian Pötzsche*
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Contents

Notation

Sets, mappings, numbers: Let X, Y be sets. The empty set is denoted by the symbol $\emptyset$, 2^X is the power set (the set of all subsets) of X, and if X is finite, $\#X$ is its cardinality. For a mapping $f : X \to Y$ the *restriction* to a subset $A \subseteq X$ is denoted by $f|_A$. If $Y \subseteq X$ we define the *iterates* $f^n : X \to Y$ of f, where n is a nonnegative integer, recursively as

$$f^0(x) := x, \qquad f^n(x) := f \circ f^{n-1}(x) \quad \text{for all } n > 0,\ x \in X.$$

As frequently used in the mathematical literature, we write

$\mathbb{N}$	for the natural numbers $\{1, 2, 3, \ldots\}$,
$\mathbb{Z}$	for the integers $\{0, \pm 1, \pm 2, \ldots\}$,
$\mathbb{R}$	for the field of real numbers,
$\mathbb{C}$	for the field of complex numbers,
$\mathbb{C}_\infty$	for the extended complex plane, and
$\mathbb{F}$	for one of the fields $\mathbb{R}$ or $\mathbb{C}$.

In the latter sets, $|\cdot|$ is the real or complex absolute value. The complex-conjugate of $\zeta \in \mathbb{C}$ is denoted by $\overline{\zeta}$; we write $\Re\zeta$ for its real and $\Im\zeta$ for its imaginary part.

For given $k, n \in \mathbb{N}$ we write

$$P_k(n) := \left\{ (N_1, \ldots, N_k) \in \left(2^{\{1,\ldots,n\}}\right)^k \,\middle|\, \begin{array}{l} N_1 \cup \ldots \cup N_k = \{1, \ldots, n\}, \\ N_i \cap N_j = \emptyset \text{ for } i \neq j \end{array} \right\}$$

for the set of all *partitions* of $\{1, \ldots, n\}$ with *length* k and indexing a nonempty subset $N_i \subseteq \{1, \ldots, n\}$ by $N_i = \left\{n_1^i, \ldots, n_{\#N_i}^i\right\}$, we define

$$P_k^<(n) := \left\{ (N_1, \ldots, N_k) \in P_k(n) \,\middle|\, \begin{array}{l} \#N_i \geq 1 \text{ for } 1 \leq i \leq k, \\ n_1^i < \ldots < n_{\#N_i}^i \text{ for } 1 \leq i \leq k, \\ \max N_i < \max N_{i+1} \text{ for } 1 \leq i < k \end{array} \right\}$$

as the set of *ordered partitions* of $\{1, \ldots, n\}$ with length k. Note that a tuple from a partition can possess the empty set as component, whereas a tuple from an ordered partition has nonempty components.

Topological spaces: We require the following definitions and terminology for a subset $A \subseteq X$ of a topological space X. The *interior* of A is $\operatorname{int}_X A$, the *closure* is $\operatorname{cl}_X A$ and $\operatorname{bd}_X A$ the *boundary*. If no confusion can arise, we drop the notational dependence on the space X and simply write $\operatorname{int} A$, $\operatorname{cl} A$ or $\operatorname{bd} A$, respectively. We follow this convention in our further notation.

Metric spaces: For a metric space X, its metric is denoted by d_X (or simply d); we sometimes write (X, d_X) to emphasize the dependence on the metric d_X. For $x \in X$ and $r > 0$ we introduce the open, pointed and closed *balls* in X, resp.,

$$\begin{aligned} &B_r(x, X) := \{y \in X : d_X(x, y) < r\}, \qquad \dot{B}_r(x, X) := B_r(x, X) \setminus \{x\}, \\ &\bar{B}_r(x, X) := \{y \in X : d_X(x, y) \le r\}; \end{aligned}$$

in absence of confusion we write $B_r(x)$, $\dot{B}_r(x)$ or $\bar{B}_r(x)$, respectively.

The *diameter* of a set $A \subseteq X$ is $\operatorname{diam}_X A := \sup_{a,b \in A} d_X(a, b)$, the *distance* of a point $a \in A$ from $B \subseteq X$ is defined by $\operatorname{dist}_X(a, B) := \inf_{b \in B} d_X(a, b)$ and the *Hausdorff separation* of A and B is given by $h_X(A, B) := \sup_{a \in A} \operatorname{dist}_X(a, B)$; with a further set $C \subseteq A$ one has the monotonicity relations

$$h_X(C, B) \le h_X(A, B), \qquad h_X(B, A) \le h_X(B, C). \tag{0.0a}$$

If $X_1, \ldots, X_n$, $n \in \mathbb{N}$, are metric spaces, then their Cartesian product $\bigtimes_{k=1}^n X_k$ is always equipped with the product metric

$$d_{X_1 \times \ldots \times X_n}((x_1, \ldots, x_n), (y_1, \ldots, y_n)) := \max_{k \in \{1, \ldots, n\}} d_{X_k}(x_k, y_k) \tag{0.0b}$$

for $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \bigtimes_{k=1}^n X_k$. In [119, p. 72, (3.20.16)] it is shown that topological properties of $X_1, \ldots, X_n$ carry over to the product $\bigtimes_{k=1}^n X_k$.

Let Y be another metric space. Then a mapping $f : X \to Y$ is called *bounded*, if it maps bounded subsets of X into bounded sets in Y, and *completely continuous*, if it is continuous and *compact* (f maps bounded subsets of X into relatively compact subsets of Y). We say a subspace $X_0 \subseteq X$ is *continuously embedded* into X, if the *embedding operator* $J : X_0 \to X$, $Jx := x$, is continuous, and we write $X_0 \hookrightarrow X$. Similarly, X_0 is *compactly embedded* into X, in symbols $X_0 \Subset X$, if $J : X_0 \to X$ is compact.

A *Lipeomorphism* is a bijective Lipschitz mapping with Lipschitz inverse. For a mapping $F : X \times Z \to Y$, where Z is a nonempty set, we define the Lipschitz constants

$$\begin{aligned} \operatorname{lip} F(\cdot, z) &:= \inf \{L \ge 0 : d_Y(F(x, z), F(\bar{x}, z)) \le L d_X(x, \bar{x}) \text{ for all } x, \bar{x} \in X\}, \\ \operatorname{lip}_1 F &:= \sup_{z \in Z} \operatorname{lip} F(\cdot, z), \end{aligned}$$

supplemented with $\inf \emptyset = \infty$. If a set Z has a metric structure, one defines $\operatorname{lip}_2 F$ analogously and proceeds correspondingly, if F depends on more variables.

Linear spaces and mappings: Linear spaces X and Y of this work are always real ($\mathbb{F} = \mathbb{R}$) or complex ($\mathbb{F} = \mathbb{C}$). An n-tuple $(x, \dots, x) \in X^n$, $n \in \mathbb{N}$, of the same vector $x \in X$ is abbreviated by x^n.

The space of linear mappings between X and Y is $\mathrm{Hom}(X, Y)$, $\mathrm{Hom}(X) := \mathrm{Hom}(X, X)$ and $\mathrm{Iso}(X, Y)$ stands for the subset of linear bijections from X to Y; $\mathrm{Iso}(X) := \mathrm{Iso}(X, X)$. The identity on X is denoted by I_X. For $S \in \mathrm{Hom}(X, Y)$, we define *kernel* (or *nullspace*) and *image* (or *range*) of S, resp.,

$$\ker S := \{x \in X : Sx = 0\}, \qquad \operatorname{im} S := \{Sx \in Y : x \in X\}.$$

The *point spectrum* of $S, T \in \mathrm{Hom}(X, Y)$ reads as

$$\sigma_p(S, T) := \{\lambda \in \mathbb{C} : \ker(S - \lambda T) \neq \{0\}\}$$

and in the special case $X = Y$ and $T = I_X$ we abbreviate $\sigma_p(S) := \sigma(S, I_X)$.

Suppose that $X_1, \dots, X_n$, $n \in \mathbb{N}$, stand for further linear spaces. The values of an n-linear mapping $T : \bigtimes_{k=1}^n X_k \to Y$ are abbreviated as $Tx_1 \cdots x_n := T(x_1, \dots, x_n)$ and we write $\mathrm{Hom}(X_1, \dots, X_n; Y)$ for the linear space of all such mappings; in addition, $\mathrm{Hom}_n(X, Y) := \mathrm{Hom}(X, \dots, X; Y)$. If all X_k are subspaces of X, then $X_1 \oplus \dots \oplus X_n$ is their *direct sum*.

Topological linear spaces: Let X, Y be topological linear spaces. Then $\mathrm{co}_X A$ is the convex hull of $A \subseteq X$, and we use the notation $\overline{\mathrm{co}}_X A$ for the closure $\mathrm{cl}_X \mathrm{co}_X A$.

We write $L(X, Y)$ for the linear space of continuous maps in $\mathrm{Hom}(X, Y)$, $L(X) := L(X, X)$ and $X^\star := L(X, \mathbb{F})$ for the *dual space* of X; the corresponding *duality pairing* is $\langle x, x^\star \rangle := x^\star(x)$. Given a homomorphism $T : X \to Y$, its *dual mapping* $T^\star : Y^\star \to X^\star$ is defined via $\langle Tx, x^\star \rangle = \langle x, T^\star x^\star \rangle$ for all $x \in X$, $x^\star \in Y^\star$. The space of continuous isomorphisms $T : X \to Y$ with inverse $T^{-1} \in L(Y, X)$ is denoted by $GL(X, Y)$. Similarly, $L(X_1, \dots, X_n; Y)$ is the linear subspace of $\mathrm{Hom}(X_1, \dots, X_n; Y)$ consisting of continuous maps; $L_n(X, Y) := L(X, \dots, X; Y)$. It is convenient to write $L_0(X, Y) := Y$.

The *spectrum* and *spectral radius* of $S, T \in L(X, Y)$, resp., are given by

$$\sigma(S, T) := \{\lambda \in \mathbb{C} : S - \lambda T \notin GL(X, Y)\}, \qquad \rho(S, T) := \sup_{\lambda \in \sigma(S,T)} |\lambda|;$$

for $X = Y$ and $T = I_X$ we abbreviate $\sigma(S) := \sigma(S, I_X)$ and $\rho(S) := \rho(S, I_X)$. Sometimes one is interested in the absolute value of the spectral points, and it is convenient to write

$$|\sigma(S, T)| := \{r \in [0, \infty) : r = |\lambda| \text{ for some } \lambda \in \sigma(S, T)\}.$$

Normed spaces: For normed linear spaces X, Y, their norm is denoted by $\|\cdot\|_X$, $\|\cdot\|_Y$, respectively, or simply $\|\cdot\|$. Then $L(X, Y)$ is a normed space w.r.t. the norm $\|S\|_{L(X,Y)} := \sup_{\|x\|_X = 1} \|Sx\|_Y$. Moreover, $L(X_1, \dots, X_n; Y)$ is a normed space canonically equipped with the norm

$$\|T\|_{L(X_1, \dots, X_n; Y)} := \sup_{\|x_1\|_{X_1}, \dots, \|x_n\|_{X_n} \leq 1} \|Tx_1 \cdots x_n\|_Y;$$

it is a Banach space, if Y is complete (cf. [295, pp. 67–68]). Higher order derivatives in normed spaces lead to spaces $L(X_1, L(X_2, \ldots, L(X_n, Y) \ldots))$ which are norm-isomorphic to $L(X_1, \ldots, X_n; Y)$ by virtue of

$$[(x_1, \ldots, x_n) \mapsto Tx_1 \cdots x_n] \mapsto \left[x_1 \mapsto [x_2 \mapsto \ldots [x_n \mapsto Tx_1 \cdots x_n] \ldots]\right]$$

(cf. [295, p. 68]). Thus, we can identify the two norm-isomorphic linear spaces $L(X_1, \ldots, X_n; Y)$ and $L(X_1, L(X_2, \ldots, L(X_n, Y) \ldots))$ from now on. When all spaces $X_1, \ldots, X_n$ are equal to X we write $L_n(X; Y) := L_n(X, \ldots, X; Y))$. With a closed subspace $X_1 \subseteq X$ and $P \in L(X_1; X)$ we define $T_P \in L_n(X_1; Y)$,

$$T_P x_1 \cdots x_n := T(Px_1, \ldots, Px_2) \quad \text{for all } x_1, \ldots, x_n \in X_1 \tag{0.0c}$$

and obtain the norm estimate (cf. [295, p. 68])

$$\|T_P\| \leq \|P\|^n \|T\| \quad \text{for all } n \in \mathbb{N}. \tag{0.0d}$$

Throughout the whole book, differentiability is always understood in the sense of Fréchet differentiability (cf. [295, p. 333]). For Banach spaces X, Y and an open $A \subseteq X$, the mapping $f : A \to Y$ is said to be of *class* C^m, $m \in \mathbb{N}$, if it is m-times continuously differentiable. The derivative of a differentiable function f is denoted by $Df : A \to L(X, Y)$. For mappings $f : (x_1, \ldots, x_n) \mapsto f(x_1, \ldots, x_n)$ depending differentiably on several variables, $D_k f$ is the partial derivative w.r.t. the k-th argument, $k \in \{1, \ldots, n\}$. Inductively one defines higher order derivatives $D^l f : A \to L_l(X, Y)$ and $D_k^l f$.

Inner product spaces: On an inner product space X we denote the inner product by $\langle \cdot, \cdot \rangle_X$ or, if there is no danger of confusion, by $\langle \cdot, \cdot \rangle$. Then X is canonically normed by $\|x\|_X := \sqrt{\langle x, x \rangle_X}$ and one has the inequality

$$\pm 2\Re \langle x, y \rangle_X \leq \|x\|_X^2 + \|y\|_X^2 \quad \text{for all } x, y \in X. \tag{0.0e}$$

Complexification: In order to use spectral theory it is often handy to work in a complex linear space. This can be achieved using the subsequent considerations. For a real linear space X the *complexification* $X_{\mathbb{C}}$ is the cartesian product $X \times X$ equipped with the component-wise addition and the scalar multiplication

$$\zeta(x, y) := (\Re\zeta x - \Im\zeta y, \Im\zeta x + \Re\zeta y) \quad \text{for all } \zeta \in \mathbb{C},\ x, y \in X.$$

Thus, $X_{\mathbb{C}}$ is a linear space over $\mathbb{C}$. We identify X with the set $X \times \{0\} \subseteq X_{\mathbb{C}}$. If a real linear space Y is given, in addition, the *complexification* of $S \in \mathrm{Hom}(X, Y)$ is the linear mapping $S_{\mathbb{C}} \in \mathrm{Hom}(X_{\mathbb{C}}, Y_{\mathbb{C}})$ defined by $S_{\mathbb{C}}(x, y) := (Sx, Sy)$ for $x, y \in X$. With normed spaces X, Y one defines the norm on $X_{\mathbb{C}}$ by

$$\|(x, y)\|_{X_{\mathbb{C}}} := \sup_{\theta \in [0, 2\pi)} \|\cos\theta\, x + \sin\theta\, y\|_X \quad \text{for all } x, y \in X$$

and obtains for $S, T \in L(X, Y)$ (cf. [9, p. 161ff]) that

$$\|S_{\mathbb{C}}\|_{L(X_{\mathbb{C}}, Y_{\mathbb{C}})} = \|S\|_{L(X,Y)}, \quad \sigma(S_{\mathbb{C}}, T_{\mathbb{C}}) = \sigma(S, T), \quad \rho(S_{\mathbb{C}}, T_{\mathbb{C}}) = \rho(S, T).$$

Function spaces: With X, Y being topological spaces, $C(X, Y)$ is the set of continuous functions between X and Y. Moreover, for Banach spaces X, Y and an appropriate set $\Omega \subseteq X$, $C^m(\Omega, Y)$ denotes the set of m-times continuously differentiable functions; in case $Y = \mathbb{F}$ we briefly write $C^m(\Omega)$ and proceed similarly with function spaces defined below. $C_b^m(\Omega, Y)$ consists of C^m-functions which are bounded together with its derivatives up to order m.

Lebesgue spaces: Let $p \in [1, \infty)$ and we assume (Ω, Σ, μ) is a positive measure space. We equip the collection $L^p(\Omega)$ of μ-measurable functions $u : \Omega \to \mathbb{F}$ satisfying $\int_\Omega |u(x)|^p \, d\mu(x) < \infty$ with its canonical norm

$$\|u\|_{L^p(\Omega)} := \left(\int_\Omega |u(x)|^p \, d\mu(x) \right)^{\frac{1}{p}},$$

and $\langle u, v \rangle_{L^2(\Omega)} := \int_\Omega u(x)\overline{v(x)} d\mu(x)$ makes $L^2(\Omega)$ a Hilbert space.

Sobolev spaces: The collection of all functions $u : \Omega \to \mathbb{F}$ whose weak derivatives $d^\alpha u$ exist and satisfy $d^\alpha u \in L^p(\Omega)$ for multi-indices α, $|\alpha| \leq m$, is denoted by $W^{m,p}(\Omega)$. It becomes a separable Banach space w.r.t. the norm

$$\|u\|_{W^{m,p}(\Omega)} := \left(\int_\Omega \sum_{|\alpha| \leq m} |d^\alpha u(x)|^p \, d\mu(x) \right)^{\frac{1}{p}},$$

and $\langle u, v \rangle_{H^m(\Omega)} := \int_\Omega \sum_{|\alpha| \leq m} u(x)\overline{v(x)} d\mu(x)$ makes $H^m(\Omega) := W^{m,2}(\Omega)$ a Hilbert space. Finally, we define (cf. [2])

$$W_0^{m,p}(\Omega) := \mathrm{cl}_{W^{m,p}(\Omega)} C_0^\infty(\Omega), \qquad H_0^m(\Omega) := W_0^{m,2}(\Omega).$$

Miscellaneous: Following an established convention, "empty" sums and products are defined as follows:

$$\sum_{n=k}^{l} x_n := 0, \qquad \prod_{n=k}^{l} x_n := 1$$

for $k > l$ and elements $x_k, \ldots, x_l$ of an additive resp., multiplicative monoid.

Notations from this monograph: In this book, we are concerned with functions defined on the integers and need an appropriate notation. A *discrete interval* $\mathbb{I}$ is the intersection of a real interval with $\mathbb{Z}$ and, in particular,

$$\mathbb{Z}_\kappa^+ := \{k \in \mathbb{Z} : \kappa \leq k\}, \qquad \mathbb{Z}_\kappa^- := \{k \in \mathbb{Z} : k \leq \kappa\},$$
$$\mathbb{I}_\kappa^+ := \mathbb{I} \cap \mathbb{Z}_\kappa^+, \qquad \mathbb{I}_\kappa^- := \mathbb{I} \cap \mathbb{Z}_\kappa^- \quad \text{for all } \kappa \in \mathbb{Z};$$

we write $I_{\mathbb{Z}} := I \cap \mathbb{Z}$ for the intersection of the integers with a real interval. Frequently it is convenient to have the notation

$$\mathbb{I}' := \begin{cases} \mathbb{I} \setminus \{\max \mathbb{I}\} & \text{if } \mathbb{I} \text{ is bounded above,} \\ \mathbb{I} & \text{else.} \end{cases}$$

Let $F(A, B)$ be an expression, e.g., a term, a formula, an equation, an inequality or a logical statement depending on "variables" A and B. Whenever convenient we use the space saving abbreviation $F(A^{\pm}, B^{\mp})$ either for $F(A^{+}, B^{-})$ or $F(A^{-}, B^{+})$.

Chapter 1	
$\mathcal{S}, \mathcal{X}, \ldots$	Nonautonomous sets, p. 2
$\mathcal{S}(k)$	k-Fiber of a nonautonomous set $\mathcal{S}$, p. 2
$\mathfrak{X}, \mathfrak{Y}, \ldots$	Union of fibers, p. 2
$\lvert\mathcal{S}(k)\rvert$	p. 3
$\mathcal{B}_{\varepsilon}(\mathcal{S})$	ε-Neighborhood of $\mathcal{S}$, p. 3
ϕ'	Forward shift, p. 3
$\varphi(k, \kappa)$	2-Parameter (semi-)group, p. 3
$\hat{\varphi}_k$	Generator of a 2-parameter (semi-)group, p. 4
$\gamma_{\mathcal{B}}^{m}$	Truncated forward orbit, p. 6
$\omega_{\mathcal{B}}$	ω-Limit set, p. 8
Chapter 2	
$\varphi(k; \kappa, \xi)$	General (forward, backward) solution, p. 51
$\underline{\phi(k,p)}, \overline{\phi(k,p)}$	Row or column notation, p. 54
$\overline{\chi_u^{\pm}(\psi)}, \chi_l^{\pm}(\psi)$	Upper and lower characteristic exponents, p. 61
$\lambda_u^{\pm}(\phi, \xi), \lambda_l^{\pm}(\phi, \xi)$	Upper and lower characteristic exponents, p. 61
$C_{r,N}$	Piecewise affine continuous functions over $[-r, 0]$, p. 71
L_N^2	Discrete Lebesgue space, p. 75, p. 178
Chapter 3	
$L_{\mathbb{J}}(g)$	Solution space with inhomogeneity g, p. 97
$\Phi(k, \kappa)$	Evolution operator, p. 98
$\hat{\Phi}_k$	Generator of a linear 2-parameter (semi-)group, p. 99
$\mathcal{U}_0^{+}$	Super-stable vector bundle, p. 99
$\Sigma_L^{+}(A, B)$	Forward Lyapunov spectrum, p. 107
P, Q	Complementary projectors, p. 115
$\mathcal{P}, \mathcal{Q}$	Invariant vector bundles, p. 115
P_m^n, Q_m^n	Complementary projectors, p. 121
$\mathcal{P}_m^n, \mathcal{Q}_m^n$	Invariant vector bundles, p. 121
$\mathfrak{X}_{\kappa,c}^{\pm}$	$c^{\pm}$-Bounded sequences, p. 124
$\lVert\cdot\rVert_{\kappa,c}^{\pm}$	(Semi-)norm on $\mathfrak{X}_{\kappa,c}^{\pm}$, p. 126
$\mathfrak{X}_{c,d}$	(c, d)-Bounded sequences, p. 124
$\lVert\cdot\rVert_{\kappa,c,d}$	Norm on $\mathfrak{X}_{c,d}$, p. 126

$\mathcal{X}_c$	c-Bounded sequences, p. 124
$\lVert\cdot\rVert_{\kappa,c}$	Norm on $\mathcal{X}_c$, p. 126
$\mathcal{X}^{\pm}_{\kappa,c,B}$	B-weighted $c^{\pm}$-bounded sequences, p. 124
$\lVert\cdot\rVert^{\pm}_{\kappa,c,B}$	(Semi-)norm on $\mathcal{X}^{\pm}_{\kappa,c,B}$, p. 126
$\mathcal{X}_{c,d,B}$	B-weighted (c,d)-bounded sequences, p. 124
$\lVert\cdot\rVert_{\kappa,c,d,B}$	Norm on $\mathcal{X}_{c,d,B}$, p. 126
$\mathcal{X}_{c,B}$	B-weighted c-bounded sequences, p. 124
$\lVert\cdot\rVert_{\kappa,c,B}$	Norm on $\mathcal{X}_{c,B}$, p. 126
$\mathcal{X}^{m,\pm}_{\kappa,c}$	$c^{\pm}$-Bounded sequences in $L_m(X_\kappa, X_k)$, p. 127
$\Sigma_f(A,B)$	Forward dichotomy spectrum, p. 130
$\Sigma(A,B)$	Dichotomy spectrum, p. 130
$\sim^+_n, \sim^-_n, \sim^j_i$	Equivalence relations, p. 138, p. 139 and p. 139
$[\cdot]^+_n, [\cdot]^-_n, [\cdot]^j_i$	Equivalence classes, p. 138, p. 139 and p. 139
$\mathcal{U}^j_i$	Intersection of invariant vector bundles, p. 123
$\mathcal{U}_s, \mathcal{U}_{cs}$	Stable and center-stable vector bundle, p. 138
$\mathcal{U}_u, \mathcal{U}_{cu}$	Unstable and center-unstable vector bundle, p. 139
$\mathcal{U}_c$	Center vector bundle, p. 139
Δ_h	Discrete Laplacian, p. 176
$\delta^{\pm}_h$	Difference operators, p. 176, p. 177
D_h, D^+_h	Forward difference operator, p. 178
H^{2s}_N	Discrete Sobolev space, p. 179
$\Sigma_B(A,B)$	Bohl spectrum, p. 182
Chapter 4	
$\bar{\Gamma}_i$	Interval for growth rates, p. 195
$\mathcal{W}^{\pm}_i$	Invariant fiber bundles, p. 200
$\mathcal{W}^j_i$	Intersections of invariant fiber bundles, p. 208
$\sim^+_i, \sim^-_i, \sim^j_i$	Equivalence relations, p. 201, p. 202 and p. 209
$[\cdot]^+_i, [\cdot]^-_i, [\cdot]^j_i$	Equivalence classes, p. 201, p. 202 and p. 209
$\mathcal{V}^{\pm}_i(\xi)$	Invariant fibers, p. 221
$\pi^{\pm}_i$	Asymptotic phases, p. 225
$\mathcal{W}^{\pm}_{\phi_*}$	Stable resp. unstable set of a solution ϕ_*, p. 257
$\mathcal{W}_s, \mathcal{W}_{cs}$	Stable and center-stable fiber bundle, p. 260
$\mathcal{W}_u, \mathcal{W}_{cu}$	Unstable and center-unstable fiber bundle, p. 260
$\mathcal{W}_c$	Center fiber bundle, p. 266
$\mathcal{X}^{\pm}_{\kappa,c}(K)$	Finite sequences, p. 280
Chapter 5	
T_k	Topological conjugation, p. 320
Ψ	Solution conjugation, p. 334
Appendix	
$a < b, a \leq b$	Order relations for sequences, p. 345
$a \ll b$	Uniform order relation for sequences, p. 345

$\lfloor b-a \rfloor$	Point-wise infimum between sequences, p. 345
$\lceil a \rceil$	Point-wise supremum of a sequence a, p. 345
$r_X(x)$	Radial retraction on normed space X, p. 364
F^ρ	Lipschitz resp. C^m-extension of F, p. 365, p. 370
χ_ρ	C^m-cut-off function, p. 369
$\operatorname{dar} f$	Darbo constant of f, p. 353