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Editors:

J.-M. Morel, Cachan

F. Takens, Groningen

B. Teissier, Paris

Hans Schoutens

The Use of Ultraproducts in Commutative Algebra

Hans Schoutens
City University of New York
CUNY Graduate Center
Department of Mathematics
365 Fifth Avenue
New York, NY, 10016
USA
hschoutens@citytech.cuny.edu

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*To my mother, Jose Van Passel,
for giving me wisdom;
to my father, Louis Schoutens,
for giving me knowledge;
to my teacher, Pierre Gevers,
for giving me the love for mathematics;
to my mentor, Jan Deneff,
for giving me inspiration;
and to my one and true love,
Parvaneh Pourshariati,
for giving me purpose.*

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