

# Lecture Notes in Mathematics

1996

**Editors:**

J.-M. Morel, Cachan

F. Takens, Groningen

B. Teissier, Paris

Thomas Lorenz

# Mutational Analysis

A Joint Framework for Cauchy Problems  
In and Beyond Vector Spaces

Thomas Lorenz  
Goethe University Frankfurt am Main  
Institute of Mathematics  
Robert-Mayer-Str. 10  
60325 Frankfurt am Main  
Germany  
lorenz@math.uni-frankfurt.de

ISBN: 978-3-642-12470-9 e-ISBN: 978-3-642-12471-6  
DOI: 10.1007/978-3-642-12471-6  
Springer Heidelberg Dordrecht London New York

Lecture Notes in Mathematics ISSN print edition: 0075-8434  
ISSN electronic edition: 1617-9692

Library of Congress Control Number: 2010928326

Mathematics Subject Classification (2000): 34A60, 34G10, 35K20, 49J53, 60H20, 93B03

© Springer-Verlag Berlin Heidelberg 2010

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilm or in any other way, and storage in data banks. Duplication of this publication or parts thereof is permitted only under the provisions of the German Copyright Law of September 9, 1965, in its current version, and permission for use must always be obtained from Springer. Violations are liable to prosecution under the German Copyright Law.

The use of general descriptive names, registered names, trademarks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

*Cover design:* SPi Publisher Services

Printed on acid-free paper

springer.com

## Preface

**Differential problems should not be restricted to vector spaces in general.**

### The Main Goal of This Book

*Ordinary differential equations* play a central role in science. Newton's Second Law of Motion relating force, mass and acceleration is a very famous and old example formulated via derivatives. The theory of ordinary differential equations was extended from the finite-dimensional Euclidean space to (possibly infinite-dimensional) Banach spaces in the course of the twentieth century. These so-called *evolution equations* are based on strongly continuous semigroups.

For many applications, however, it is difficult to specify a suitable normed vector space. Shapes, for example, do not have an obvious linear structure if we dispense with any a priori assumptions about regularity and thus, we would like to describe them merely as compact subsets of the Euclidean space.

*Hence, this book generalizes the classical theory of ordinary differential equations beyond the borders of vector spaces.* It focuses on the well-posed Cauchy problem in any finite time interval.

In other words, states are evolving in a set (not necessarily a vector space) and, they determine their own evolution according to a given “rule” concerning their current “rate of change” — a form of feedback (possibly even with finite delay). In particular, the examples here do not have to be gradient systems in metric spaces.

### The Driving Force of Generalization: Solutions via Euler Method

The step-by-step extension starts in metric spaces and ends up in nonempty sets that are merely supplied with suitable families of distance functions (not necessarily symmetric or satisfying the triangle inequality).

Solutions to the abstract Cauchy problem are usually constructed by means of the Euler method and so the key question for each step of conceptual generalization is: Which aspect of the a priori given structures can be still weakened so that the Euler method does not fail ?

### Diverse Examples Have Always Given Directions ... Towards a Joint Framework.

In the 1990s, Jean-Pierre Aubin suggested what he called *mutational equations* and applied them to systems of ordinary differential equations and time-dependent compact subsets of  $\mathbb{R}^N$  (equipped with the popular Pompeiu-Hausdorff metric). They are the starting point of this monograph.

Further examples, however, reveal that Aubin's a priori assumptions (about the additional structure of the metric space) are quite restrictive indeed. There is no obvious way for applying the original theory to semilinear evolution equations.

Our basic strategy to generalize mutational equations is simple: Consider several diverse examples successively and, whenever it does not fit in the respective mutational framework, then find some extension for overcoming this obstacle.

Mutational Analysis is definitely not just to establish another abstract term of solution though. Hence, it is an important step to check for each example individually whether there are relations to some more popular meaning (like classical, strong, weak or mild solution).

Here are some of the examples under consideration in this book:

- Feedback evolutions of nonempty compact subsets of  $\mathbb{R}^N$   
Application to image segmentation
- Birth-and-growth processes of random closed sets (not necessarily convex)
- Semilinear evolution equations in arbitrary Banach spaces
- Nonlocal parabolic differential equations in noncylindrical domains
- Nonlinear transport equations for Radon measures on  $\mathbb{R}^N$
- Structured population model with Radon measures on  $\mathbb{R}_0^+$
- Stochastic ordinary differential equations with nonlocal sample dependence

In particular, these examples can now be coupled in systems immediately – due to the *joint* framework of Mutational Analysis. This possibility provides new tools for modelling in future.

### **The Structure of This Extended Book ... for the Sake of the Reader**

This monograph is written as a synthesis of two aims: first, the reader should have quick access to the results of individual interest and second, all mathematical conclusions are presented in detail so that they are sufficiently comprehensible.

Each chapter is elaborated in a quite self-contained way so that the reader has the opportunity to select freely according to the examples of personal interest. Hence some arguments typical for mutational analysis might appear rather frequently, but they are always adapted to the respective framework. Moreover, the proofs are usually collected at the end of each subsection so that they can be skipped easily if wanted. References to results elsewhere in the monograph are usually supplied with page numbers. Each example contains a table that summarizes the choice of basic sets, distances etc. and indicates where to find the main results.

The introductory Chapter 0 summarizes the essential notions and motivates the generalizations in this book. Many of the subsequent conclusions have their origins in §§ 1.1 – 1.6 and so these subsections facilitate understanding the modifications later.

Experience has already taught that such a monograph cannot be written free from any errors or mistakes. I would like to apologize in advance and hope that the gist of both the approach and examples is clear. Comments are very welcome.

## Acknowledgments

This monograph would not have been completed if I had not benefited from the harmony and the support in my vicinity. Both the scientific and the private aspect are closely related in this context.

Prof. Willi Jäger has been my academic teacher since my very first semester at Heidelberg University. Infected by the “virus” of analysis, I followed his courses, full of insights into mathematical relations. As a part of his scientific support, he drew my attention to set-valued maps quite early and gave me the opportunity to gain experience very autonomously. I would like to express my deep gratitude to Prof. Jäger.

Moreover, I am deeply indebted to Prof. Jean–Pierre Aubin and H       Frankowska. Their mathematical influence on me started quite early — as a consequence of their monographs. During three stays at CREA of Ecole Polytechnique in Paris, I benefited from collaborating with them and meeting several colleagues sharing my mathematical interests partly.

Furthermore, I would like to thank all my friends, collaborators and colleagues for the inspiring discussions and observations over time. This list (in alphabetical order) is neither complete nor a representative sample, of course: Zvi Artstein, Robert Baier, Bruno Becker, Hans Belzer, Christel Br      , Eva Cr      , Roland Dinkel, Herbert-Werner Diskut, Tzanko Donchev, Matthias Gerdts, Piotr Gwiazda, Peter E. Kloeden, Roger K      , Stephan Luckhaus, Anna Marciniak-Czochra, Reinhard Mohr, Jerzy Motyl, Jos   Alberto Murillo Hern      , Janosch Rieger, Ina Scheid, Ursula Schmitt, Roland Schnaubelt, Oliver Schn      , Jens Starke, Angela Stevens, Martha Stocker, Christina Surulescu, Manfred Taufertsh      , Friedrich Tomi, Edelgard Wei        , Kurt Wolber.

Heidelberg University and, in particular, its Interdisciplinary Center for Scientific Computing (IWR) were extraordinary home institutions for me until my habilitation (for *venia legendi* in mathematics) in 2009. In addition, some results of this monograph were elaborated as parts of projects or during research stays funded by

- German Research Foundation DFG (SFB 359 and LO 273)
- Hausdorff Research Institute for Mathematics in Bonn (spring 2008)
- Research Training Network “Evolution Equations for Deterministic and Stochastic Systems” (HPRN–CT–2002–00281) of the European Community
- Minerva Foundation for scientific cooperation between Germany and Israel.

Finally, I would like to express my deep gratitude to my family.

My parents have always supported me and have provided a harmonic environment so that I have been able to concentrate on my studies. I surely would not have reached my current situation without them as a permanent pillar.

Meanwhile my wife Irina Surovtsova has been at my side for several years. I have always trusted her to give me good advice and so she has often enabled me to overcome obstacles — both in everyday life and in science. I am optimistic that together we can cope with the challenges that Daniel, Michael and the “other aspects” of life provide for us.

*J.L*

# Contents

Preface .....	v
Acknowledgments .....	vii
<b>0 Introduction</b> .....	1
0.1 Diverse Evolutions Come Together Under the Same Roof .....	1
0.2 Some Introductory Examples .....	3
0.2.1 A Region Growing Method of Image Segmentation .....	3
0.2.2 Image Smoothing via Anisotropic Diffusion .....	8
0.2.3 A Stochastic Differential Game without Precisely Known Realizations of Opponents .....	11
0.3 Extending the Traditional Horizon: Evolution Equations Beyond Vector Spaces .....	12
0.3.1 Aubin's Initial Notion: Regard Affine Linear Maps Just as a Special Type of "Elementary Deformations". ....	12
0.3.2 Mutational Analysis as an "Adaptive Black Box" for Initial Value Problems .....	15
0.3.3 The Initial Problem Decomposition and the Final Link to More Popular Meanings of Abstract Solutions .....	17
0.3.4 The New Steps of Generalization .....	18
0.4 Mutational Inclusions .....	29
<b>1 Extending Ordinary Differential Equations to Metric Spaces: Aubin's Suggestion</b> .....	31
1.1 The Key for Avoiding (Affine) Linear Structures: Transitions .....	31
1.2 The Mutation as Counterpart of Time Derivative .....	37
1.3 Feedback Leads to Mutational Equations .....	38
1.4 Proofs for Existence and Uniqueness of Solutions without State Constraints .....	40
1.5 An Essential Advantage of Mutational Equations: Solutions to Systems .....	44
1.6 Proof for Existence of Solutions Under State Constraints .....	47
1.7 Some Elementary Properties of the Contingent Transition Set .....	51
1.8 Example: Ordinary Differential Equations in $\mathbb{R}^N$ .....	53

1.9	Example: Morphological Equations for Compact Sets in $\mathbb{R}^N$ . . . . .	57
1.9.1	The Pompeiu-Hausdorff Distance $d$ . . . . .	57
1.9.2	Morphological Transitions on $(\mathcal{K}(\mathbb{R}^N), d)$ . . . . .	60
1.9.3	Morphological Primitives as Reachable Sets . . . . .	64
1.9.4	Some Examples of Morphological Primitives . . . . .	66
1.9.5	Some Examples of Contingent Transition Sets . . . . .	67
1.9.6	Solutions to Morphological Equations . . . . .	74
1.10	Example: Morphological Transitions for Image Segmentation . . . . .	79
1.10.1	Analytical Tools of the Continuous Segmentation Problem . . . . .	80
1.10.2	Solving the Continuous Segmentation Problem . . . . .	83
1.10.3	The Application to Computer Images . . . . .	90
1.11	Example: Modified Morphological Equations via Bounded One-Sided Lipschitz Maps . . . . .	96
<b>2</b>	<b>Adapting Mutational Equations to Examples in Vector Spaces . . . . .</b>	<b>103</b>
2.1	The Topological Environment of This Chapter . . . . .	104
2.2	Specifying Transitions and Mutation on $(E, (d_j)_{j \in \mathcal{J}}, ([\cdot]_j)_{j \in \mathcal{J}})$ . . . . .	104
2.3	Solutions to Mutational Equations . . . . .	107
2.3.1	Continuity with Respect to Initial States and Right-Hand Side . . . . .	108
2.3.2	Limits of Pointwise Converging Solutions: Convergence Theorem . . . . .	109
2.3.3	Existence for Mutational Equations without State Constraints . . . . .	112
2.3.4	Convergence Theorem and Existence for Systems . . . . .	116
2.3.5	Existence for Mutational Equations with Delay . . . . .	120
2.3.6	Existence Under State Constraints for Finite Index Set $\mathcal{J}$ . . . . .	123
2.4	Example: Semilinear Evolution Equations in Reflexive Banach Spaces . . . . .	125
2.5	Example: Nonlinear Transport Equations for Radon Measures on $\mathbb{R}^N$ . . . . .	132
2.5.1	The $W^{1,\infty}$ Dual Metric $\rho_{\mathcal{M}}$ on Radon Measures $\mathcal{M}(\mathbb{R}^N)$ . . . . .	132
2.5.2	Linear Transport Equations Induce Transitions on $\mathcal{M}(\mathbb{R}^N)$ . . . . .	136
2.5.3	Conclusions About Nonlinear Transport Equations . . . . .	142
2.6	Example: A Structured Population Model with Radon Measures over $\mathbb{R}_0^+ = [0, \infty[$ . . . . .	147
2.6.1	Introduction . . . . .	147
2.6.2	The Linear Population Model . . . . .	152
2.6.3	Conclusions About the Full Nonlinear Population Model . . . . .	164
2.7	Example: Modified Morphol. Equations via One-Sided Lipschitz Maps . . . . .	171



<b>3</b>	<b>Continuity of Distances Replaces the Triangle Inequality</b>	<b>181</b>
3.1	General Assumptions of This Chapter	182
3.2	The Essential Features of Transitions Do Not Change	185
3.3	Solutions to Mutational Equations	186
3.3.1	Continuity with Respect to Initial States and Right-Hand Side	189
3.3.2	Limits of Graphically Converging Solutions: Convergence Theorem	190
3.3.3	Existence for Mutational Equations with Delay and without State Constraints	193
3.3.4	Existence for Systems of Mutational Equations with Delay	198
3.3.5	Existence Under State Constraints for a Single Index	202
3.3.6	Exploiting a Generalized Form of “Weak” Compactness: Convergence and Existence without State Constraints	206
3.3.7	Existence of Solutions due to Completeness: Extending the Cauchy-Lipschitz Theorem	212
3.4	Local $\omega$ -Contractivity of Transitions Can Become Dispensable	214
3.5	Considering Tuples with a Separate Real Time Component	221
3.6	Example: Strong Solutions to Nonlocal Stochastic Differential Equations	231
3.6.1	The General Assumptions for This Example	233
3.6.2	Some Standard Results About Itô Integrals and Strong Solutions to Stochastic Ordinary Differential Equations	233
3.6.3	A Short Cut to Existence of Strong Solutions	235
3.6.4	A Special Case with Fixed Additive Noise in More Detail	238
3.7	Example: Stochastic Morphological Equations for Square Integrable Random Closed Sets in $\mathbb{R}^N$	242
3.7.1	The General Assumptions for This Example	243
3.7.2	Reachable Sets of Stochastic Differential Inclusions are to Induce Transitions on $\mathcal{H}^2(\Omega, \mathbb{R}^N)$	247
3.7.3	The Main Conclusions About Stochastic Growth Processes	251
3.7.4	Extensions to Stochastic Birth-and-Growth Processes	254
3.8	Example: Nonlinear Continuity Equations with Coefficients of BV for $\mathcal{L}^N$ Measures	260
3.8.1	The Lagrangian Flow in the Sense of Ambrosio	262
3.8.2	The Subset $\mathbb{L}^{\infty \cap 1}(\mathbb{R}^N)$ of Measures and its Pseudo-Metrics	264
3.8.3	Autonomous Linear Continuity Problems Induce Transitions on $\mathbb{L}^{\infty \cap 1}(\mathbb{R}^N)$ via Lagrangian Flows	266
3.8.4	Conclusions About Nonlinear Continuity Equations	272

3.9	Example: Nonlocal Parabolic Equations in Cylindrical Domains . . .	278
3.9.1	Motivation: Smoothing an Image, but Preserving its Edges .	278
3.9.2	The Main Result . . . . .	279
3.9.3	The Underlying Details in Terms of Mutational Analysis . .	282
3.10	Example: Semilinear Evolution Equations in Any Banach Spaces . .	291
3.10.1	The Distance Functions $(d_j)_{j \in \mathbb{R}_0^+}, (\tilde{e}_j)_{j \in \mathbb{R}_0^+}$ on $\tilde{X} = \mathbb{R} \times X$ . .	293
3.10.2	The Variation of Constants Induces Transitions on $\tilde{X}$ . . . . .	298
3.10.3	Mild Solutions to Semilinear Evolution Equations in $X$ — Using an Immediately Compact Semigroup . . . . .	300
3.10.4	Exploiting Relatively Compact Terms of Inhomogeneity . .	306
3.11	Example: Parabolic Differential Equations in Noncylindrical Domains . . . . .	311
3.11.1	The General Assumptions for This Example . . . . .	311
3.11.2	Some Results of Lumer and Schnaubelt . . . . .	313
3.11.3	Semilinear Parabolic Differential Equations in a Fixed Noncylindrical Domain . . . . .	317
3.11.4	The Tusk Condition for Approximative Cauchy Barriers . .	326
3.11.5	Successive Coupling of Nonlinear Parabolic Problem and Morphological Equation . . . . .	329
<b>4</b>	<b>Introducing Distribution-Like Solutions to Mutational Equations . . .</b>	<b>331</b>
4.1	General Assumptions of This Chapter . . . . .	334
4.2	Comparing with “Test Elements” of $\tilde{\mathcal{D}}$ along Timed Transitions . .	338
4.3	Timed Solutions to Mutational Equations . . . . .	339
4.3.1	Continuity with Respect to Initial States and Right-Hand Side . . . . .	341
4.3.2	Convergence of Timed Solutions . . . . .	345
4.3.3	Existence for Mutational Equations with Delay and without State Constraints . . . . .	348
4.3.4	Existence of Timed Solutions without State Constraints due to Another Form of “Weak” Euler Compactness . . . . .	353
4.4	Example: Mutational Equations for Compact Sets in $\mathbb{R}^N$ Depending on Normal Cones . . . . .	359
4.4.1	Limiting Normal Cones Induce Distance $d_{\mathcal{K},N}$ on $\mathcal{K}(\mathbb{R}^N)$ . .	359
4.4.2	Reachable Sets of Differential Inclusions Provide Transitions . . . . .	360
4.4.3	Existence of Solutions due to Transitional Euler Compactness . . . . .	368

4.5	Further Example: Mutational Equations for Compact Sets Depending on Normal Cones . . . . .	372
4.5.1	Specifying Sets and Distance Functions . . . . .	373
4.5.2	Reachable Sets Induce Timed Transitions . . . . .	376
4.5.3	Existence due to Strong-Weak Transitional Euler Compactness . . . . .	381
4.5.4	Uniqueness of Timed Solutions . . . . .	383
<b>5</b>	<b>Mutational Inclusions in Metric Spaces . . . . .</b>	<b>385</b>
5.1	Mutational Inclusions without State Constraints . . . . .	386
5.1.1	Solutions to Mutational Inclusions: Definition and Existence	386
5.1.2	A Selection Principle Generalizing the Theorem of Antosiewicz-Cellina . . . . .	388
5.1.3	Proofs on the Way to Existence Theorem 5.4 . . . . .	395
5.2	Morphological Inclusions with State Constraints: A Viability Theorem . . . . .	399
5.2.1	(Well-Known) Viability Theorem for Differential Inclusions	400
5.2.2	Adapting This Concept to Morphological Inclusions: The Main Theorem . . . . .	401
5.2.3	The Steps for Proving the Morphological Viability Theorem	403
5.3	Morphological Control Problems for Compact Sets in $\mathbb{R}^N$ with State Constraints . . . . .	413
5.3.1	Formulation . . . . .	415
5.3.2	The Link to Morphological Inclusions . . . . .	416
5.3.3	Application to Control Problems with State Constraints . . . .	418
5.3.4	Relaxed Control Problems with State Constraints . . . . .	420
5.3.5	Clarke Tangent Cone in the Morphological Framework: The Circatangent Transition Set . . . . .	426
5.3.6	The Hypertangent Transition Set . . . . .	433
5.3.7	Closed Control Loops for Problems with State Constraints . .	436
	<b>Tools . . . . .</b>	<b>439</b>
A.1	The Lemma of Gronwall and its Generalizations . . . . .	439
A.2	Filippov's Theorem for Differential Inclusions . . . . .	443
A.3	Scorza-Dragoni Theorem and Applications to Reachable Sets . . . .	446
A.4	Relaxation Theorem of Filippov-Ważewski for Differential Inclusions . . . . .	452

A.5	Regularity of Reachable Sets of Differential Inclusions . . . . .	454
A.5.1	Normal Cones and Compact Sets: Definitions and Notation .	454
A.5.2	Adjoint Arcs for Evolving Normal Cones to Reachable Sets	456
A.5.3	Hamiltonian System Helps Preserving $C^{1,1}$ Boundaries . . . .	458
A.5.4	How to Guarantee Reversibility of Reachable Sets in Time .	461
A.5.5	How to Make Points Evolve into Convex Sets of Positive Erosion . . . . .	463
A.5.6	Reachable Sets of Balls and Their Complements . . . . .	469
A.5.7	The (Uniform) Tusk Condition for Graphs of Reachable Sets	473
A.6	Reynolds Transport Theorem for Differential Inclusions with Carathéodory Maps . . . . .	476
A.7	Differential Inclusions with One-Sided Lipschitz Continuous Maps .	480
A.8	Stochastic Differential Inclusions in $\mathbb{R}^N$ . . . . .	482
A.8.1	Filippov-Like Theorem of Da Prato and Frankowska . . . . .	482
A.8.2	A Sufficient Condition on Invariant Subsets . . . . .	486
A.9	Proximal Normals of Set Sequences in $\mathbb{R}^N$ . . . . .	487
A.10	Tools for Set-Valued Maps . . . . .	489
A.10.1	Measurable Set-Valued Maps . . . . .	489
A.10.2	Parameterization of Set-Valued Maps . . . . .	491
A.11	Compactness of Continuous Functions Between Metric Spaces . . . .	491
A.12	Bochner Integrals and Weak Compactness in $L^1$ . . . . .	492
	<b>Bibliographical Notes</b> . . . . .	493
	References . . . . .	497
	Index of Notation . . . . .	505
	Index . . . . .	506