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Thomas Lorenz

Mutational Analysis

A Joint Framework for Cauchy Problems In and Beyond Vector Spaces

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Preface

Differential problems should not be restricted to vector spaces in general.

The Main Goal of This Book

Ordinary differential equations play a central role in science. Newton's Second Law of Motion relating force, mass and acceleration is a very famous and old example formulated via derivatives. The theory of ordinary differential equations was extended from the finite-dimensional Euclidean space to (possibly infinite-dimensional) Banach spaces in the course of the twentieth century. These so-called *evolution equations* are based on strongly continuous semigroups.
For many applications, however, it is difficult to specify a suitable normed vector space. Shapes, for example, do not have an obvious linear structure if we dispense with any a priori assumptions about regularity and thus, we would like to describe them merely as compact subsets of the Euclidean space.

Hence, this book generalizes the classical theory of ordinary differential equations beyond the borders of vector spaces. It focuses on the well-posed Cauchy problem in any finite time interval.

In other words, states are evolving in a set (not necessarily a vector space) and, they determine their own evolution according to a given "rule" concerning their current "rate of change" — a form of feedback (possibly even with finite delay). In particular, the examples here do not have to be gradient systems in metric spaces.

The Driving Force of Generalization: Solutions via Euler Method

The step-by-step extension starts in metric spaces and ends up in nonempty sets that are merely supplied with suitable families of distance functions (not necessarily symmetric or satisfying the triangle inequality).
Solutions to the abstract Cauchy problem are usually constructed by means of the Euler method and so the key question for each step of conceptual generalization is: Which aspect of the a priori given structures can be still weakened so that the Euler method does not fail ?

Diverse Examples Have Always Given Directions ... Towards a Joint Framework.

In the 1990s, Jean-Pierre Aubin suggested what he called *mutational equations* and applied them to systems of ordinary differential equations and time-dependent compact subsets of $\mathbb{R}^N$ (equipped with the popular Pompeiu-Hausdorff metric). They are the starting point of this monograph.
Further examples, however, reveal that Aubin's a priori assumptions (about the additional structure of the metric space) are quite restrictive indeed. There is no obvious way for applying the original theory to semilinear evolution equations.

Our basic strategy to generalize mutational equations is simple: Consider several diverse examples successively and, whenever it does not fit in the respective mutational framework, then find some extension for overcoming this obstacle.

Mutational Analysis is definitely not just to establish another abstract term of solution though. Hence, it is an important step to check for each example individually whether there are relations to some more popular meaning (like classical, strong, weak or mild solution).

Here are some of the examples under consideration in this book:

- Feedback evolutions of nonempty compact subsets of $\mathbb{R}^N$
  Application to image segmentation
- Birth-and-growth processes of random closed sets (not necessarily convex)
- Semilinear evolution equations in arbitrary Banach spaces
- Nonlocal parabolic differential equations in noncylindrical domains
- Nonlinear transport equations for Radon measures on $\mathbb{R}^N$
- Structured population model with Radon measures on $\mathbb{R}_0^+$
- Stochastic ordinary differential equations with nonlocal sample dependence

In particular, these examples can now be coupled in systems immediately – due to the *joint* framework of Mutational Analysis. This possibility provides new tools for modelling in future.

The Structure of This Extended Book ... for the Sake of the Reader

This monograph is written as a synthesis of two aims: first, the reader should have quick access to the results of individual interest and second, all mathematical conclusions are presented in detail so that they are sufficiently comprehensible.

Each chapter is elaborated in a quite self-contained way so that the reader has the opportunity to select freely according to the examples of personal interest. Hence some arguments typical for mutational analysis might appear rather frequently, but they are always adapted to the respective framework. Moreover, the proofs are usually collected at the end of each subsection so that they can be skipped easily if wanted. References to results elsewhere in the monograph are usually supplied with page numbers. Each example contains a table that summarizes the choice of basic sets, distances etc. and indicates where to find the main results.

The introductory Chapter 0 summarizes the essential notions and motivates the generalizations in this book. Many of the subsequent conclusions have their origins in §§ 1.1 – 1.6 and so these subsections facilitate understanding the modifications later.

Experience has already taught that such a monograph cannot be written free from any errors or mistakes. I would like to apologize in advance and hope that the gist of both the approach and examples is clear. Comments are very welcome.

Heidelberg and Frankfurt, January 2010 Thomas Lorenz

Acknowledgments

This monograph would not have been completed if I had not benefited from the harmony and the support in my vicinity. Both the scientific and the private aspect are closely related in this context.

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