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Generalized Bessel Functions of the First Kind

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Dedicated to my children Boróka and Koppány

Preface

Bessel functions are indispensable in many branches of mathematics and applied mathematics. Thus, it is important to study their properties in many aspects. Recently, there has been a vivid interest on Bessel and hypergeometric functions from the point of view of geometric function theory and functional inequalities. Although many inequalities and geometric properties of Bessel and hypergeometric functions appear in works of many mathematicians, there is no unified treatment of the topic. I have written this monograph in order to partially fill this gap in the literature. The major part of this monograph is taken from my Ph.D. thesis [54] with the same title as this monograph and that is why most results are due to myself and my coauthors. The literature has grown very quickly during the past few years and everything could not have been covered. I have tried to follow closely the structure of my thesis.

Most of my papers used in this monograph were supported partially by the Institute of Mathematics, University of Debrecen, Hungary and some of these papers were completed during my visit to University of Debrecen. I take this opportunity to thank this institution for its excellent research facilities and to thank Péter T. Nagy for his constant encouragement during the course of my work. My research was also partially supported by the János Bolyai Research Scholarship of the Hungarian Academy of Sciences since September 2009. I would also like to thank this Institution for the financial support.

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Notation

Following is a list of notation used in this book:

$\mathbb{N}, \mathbb{R}, \mathbb{C}$	Set of natural, real and complex numbers
$\mathbb{D}$	Open unit disk in the complex plane
$(a)_n$	Pochhammer (or Appell) symbol
$F(a,b,c,z)$	Gaussian hypergeometric function
$\mathscr{K}(x)$	Complete elliptic integral of the first kind
$\Phi(a,c,z)$	Kummer hypergeometric function
${}_qF_r(a_1,\ldots,a_q,b_1,\ldots,b_r,z)$	Generalized hypergeometric function
J_p, I_p	Bessel and modified Bessel function of the first kind of order p
j_p, i_p	Spherical and modified spherical Bessel function of the first kind of order p
w_p	Generalized Bessel function of the first kind of order p
u_p	Generalized and normalized Bessel function of the first kind of order p
$\Gamma(z)$	Euler's gamma function
$B(p,q)$	Euler's beta function
log	Natural logarithm function on $(0,\infty)$
Log	Principal branch of the logarithm function
γ	Euler-Mascheroni constant
$\mathscr{S}, \mathscr{S}^*, \mathscr{C}$	Class of univalent, starlike and convex functions
$\mathscr{S}^*(\alpha), \mathscr{C}(\alpha)$	Class of starlike and convex functions of order α
$\mathscr{H}^\mu$	Hardy space of analytic functions
$*$	Hadamard product $(f*g)$ of two power series
$\sim$	Asymptotic to $(f(x)\sim g(x)$ as $x\to a$ if $\lim_{x\to a} f(x)/g(x)=1)$
$\mathscr{O}(r)$	Landau order symbol
$\sin, \cos, \sinh, \cosh$	Sine, cosine, hyperbolic sine, hyperbolic cosine

$\operatorname{Re} z, \operatorname{Im} z$	Real and imaginary part of the complex number z
$A(r,s), G(r,s), H(r,s)$	Arithmetic, geometric and harmonic mean of r and s
$A_2(r,s)$	Second-order power mean of r and s
$H_p(r,s)$	p-order power mean (or Hölder mean) of r and s
$j_{p,n}$	nth positive zero of the Bessel function J_p
$\operatorname{Si}(x), \operatorname{Shi}(x)$	Sine and hyperbolic sine integral

Survey

The aim of this brief survey is to give the reader a short overview of the topics discussed in this book.

Special functions. Special functions are some particular mathematical functions which have more or less established names and notations due to their importance in mathematical analysis, functional analysis, physics, or other applications. However, there is no general formal definition, but the list of mathematical functions contains functions which are commonly accepted as special. In particular, elementary functions, especially trigonometric functions are also considered as special functions. The theory of special functions has been developed essentially in the nineteenth century due to the contributions of C.F. Gauss, C.G.J. Jacobi, F. Klein, and many others. Because of their remarkable properties, special functions have been used for centuries. For example, since they have numerous applications in astronomy, trigonometric functions have been studied for over a thousand years. However, in the twentieth century the theory of special functions has been overshadowed by other fields such as real and functional analysis, topology, algebra and differential equations. All the same, in that times appeared G.N. Watson's book [227] entitled "A treatise on the theory of Bessel functions," which is a very important book in the theory of special functions, especially in the asymptotic expansions of Bessel functions. Nowadays this book is a classic and because of their remarkable properties special functions, like Bessel and hypergeometric functions are frequently used in probability, statistics, mathematical physics and engineering sciences. This is why the Hungarian mathematician Paul Turán, as Richard Askey said, considered "special functions" a misnomer: they should be called useful functions.

Special functions in geometric function theory. Special functions play an important role in geometric function theory. Maybe the most known application is the solution of the famous Bieberbach conjecture by L. de Branges. In the proceedings of the 1986 meeting to celebrate the proof of L. de Branges' theorem, L.V. Ahlfors [4] wrote:

> To my knowledge, de Branges may be the first who has tapped the rich reservoir of knowledge hidden in the volumes on special functions and sometimes relegated to a corner of the library, and applied it to the coefficient problem. In this case, what he used was original work by contemporary mathematicians, but it raises the question whether special functions, as a powerful tool, have not been unduly neglected.

The surprising use of generalized hypergeometric functions by L. de Branges has generated considerable interest, and the geometric properties of the generalized, Gauss and Kummer hypergeometric functions have been investigated by many authors in the last few decades. Although the geometric properties of these functions are interesting in their own right, they have shown to be useful in many other problems in geometric function theory. For example, recently the starlikeness of the normalized Gaussian hypergeometric functions was used by J.H. Choi et al. [82] in order to deduce sharp norm estimates for the Alexander transforms of convex functions of order alpha. The investigation of the geometric properties of Bessel functions – discussed in details in this book – was motivated by the immense research about hypergeometric functions and some further researches are in progress.

Functional inequalities involving special functions. In 1991 M. Hazewinkel, as the editor of Kluwer Academic Publishers (Dordrecht/Boston/London) wrote:

> Inequalities are everywhere. Whole series of conferences are devoted to them. Indeed in my more despondent moments, when struggling with one or another problem, I sometimes have the feeling that mathematics (especially analysis) is all inequalities.

The functional inequalities which involve hypergeometric functions, especially elliptic integrals are very useful in quasiconformal analysis, and many of them have been deduced to solve some interesting problems in this topic. A large number of such inequalities has been collected by G.D. Anderson et al. in [19], which is the first book containing a systematic and detailed treatment of Gaussian hypergeometric functions from the point of view of inequalities. Some of these inequalities have been motivated naturally by the beautiful identities of elliptic integrals and nowadays the research of such inequalities is in active progress.

Since the Bessel and modified Bessel functions can be viewed as generalizations of sine, cosine, hyperbolic sine and hyperbolic cosine functions, it is natural to ask whether many identities and inequalities involving these trigonometric functions can be extended to Bessel functions. Although, this problem is interesting in its own right, recently it has been shown that such inequalities can be useful in engineering sciences, such as information and communication theory (see for example the papers of Á. Baricz and Y. Sun [56, 61]).

Topics discussed in this book. In this book our aim is to study the generalized Bessel functions of the first kind from the point of view of the complex analysis on the one hand, and on the other hand from the point of view of the classical analysis. The study of these functions from the point of view of the complex analysis is mostly related to univalence, starlikeness, convexity, close-to-convexity and subordination property, while the study from the point of view of the classical analysis is related to the extensions of some trigonometric inequalities, like Mitrinović, Mahajan, Redheffer, Cusa, Jordan, Grünbaum, Landen, van der Corput, Askey, to generalized Bessel functions of the first kind. Moreover, we deduce some chain of inequalities for Gaussian and Kummer hypergeometric functions, generalized Bessel functions, modified Bessel functions, general power series with positive coefficients.