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Banach Spaces and Descriptive Set Theory: Selected Topics

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Preface

These notes are devoted to the study of some classical problems in the Geometry of Banach spaces. The novelty lies in the fact that their solution relies heavily on techniques coming from Descriptive Set Theory.

The central theme is universality problems. In particular, the text provides an exposition of the methods developed recently in order to treat questions of the following type:

(Q) Let $\mathcal{C}$ be a class of separable Banach spaces such that every space X in the class $\mathcal{C}$ has a certain property, say property (P). When can we find a separable Banach space Y which has property (P) and contains an isomorphic copy of every member of $\mathcal{C}$?

We will consider quite classical properties of Banach spaces, such as "being reflexive," "having separable dual," "not containing an isomorphic copy of c_0," "being non-universal," etc.

It turns out that a positive answer to problem **(Q)**, for any of the above mentioned properties, is possible if (and essentially only if) the class $\mathcal{C}$ is "simple." The "simplicity" of $\mathcal{C}$ is measured in set theoretic terms. Precisely, if the class $\mathcal{C}$ is analytic in a natural "coding" of separable Banach spaces, then we can indeed find a separable space Y which is universal for the class $\mathcal{C}$ and satisfies the requirements imposed above.

The text is addressed to both Functional Analysts and Set Theorists. We have tried to follow the terminology and notation employed by these two groups of researchers. Concerning Banach Space Theory, we follow the conventions adopted in the monograph of Lindenstrauss and Tzafriri [LT]. Our descriptive set theoretic terminology follows the one employed in the book of Kechris [Ke]. Still, we had to make a compromise; so throughout these notes by $\mathbb{N} = \{0, 1, 2, \dots\}$ we shall denote the natural numbers.

We proceed to discuss how this work is organized. It is divided into three parts which are largely independent from each other and can be read separately. In the first part, consisting of Chaps. 1 and 2, we display the necessary background and set up the frame in which this work will be completed. The second part, consisting of Chaps. 3 and 4, is devoted to the study of two

"gluing" techniques for producing separable Banach spaces from given classes of Banach spaces with a Schauder basis. In the third part, consisting of Chaps. 5 and 6, we present two important embedding results and their parameterized versions.

The previous material is used in Chap. 7 which is, somehow, the goal of these notes. The notion of a *strongly bounded class* of separable Banach spaces is the central concept in Chap. 7. Several natural classes of separable Banach spaces are shown to be strongly bounded. This structural information is used to answer a number of universality problems in a unified manner.

To facilitate the interested reader we have also included four appendices. The first one contains an introduction to Rank Theory, a basic theme in Descriptive Set Theory which is crucial throughout this work. In the second appendix we present some basic concepts and results from Banach Space Theory. Beside [LT], these topics are covered in great detail in other excellent books, such as [AK,Di], as well as, in the two volumes of the *Handbook of the Geometry of Banach spaces* [JL1,JL2]. In the third appendix we give a short description of a rather technical (yet very efficient) method in Descriptive Set Theory, known as the "Kuratowski–Tarski algorithm." The method is used to compute the complexity of sets and relations. Finally, in the fourth appendix we discuss some open problems.

A significant part of the material presented in these notes has been discovered jointly with S.A. Argyros and has been published in [AD]. Actually, this text is the natural sequel of [AD] since it is mainly focused on further discoveries contained in [D3] and in our joint papers with V. Ferenczi [DF] and with J. Lopez-Abad [DL]. Several new results are also included. Needless to say that the solutions of the main problems are based on the work of many researchers including, among others, B. Bossard, J. Bourgain, N. Ghoussoub, B. Maurey, G. Pisier, W. Schachermayer, and M. Zippin. Bibliographical information on the content of each chapter is contained in its final section, named as "Comments and Remarks."

We think that this work has, mainly, two reasons of interest. The first one is that it answers some basic problems in the Geometry of Banach spaces and, more important, it explains several phenomena discovered so far. The second reason is that the solutions of the relevant problems combine techniques coming from two (seemingly) unrelated disciplines; namely from Banach Space Theory and from Descriptive Set Theory.

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