

# Lecture Notes in Mathematics

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# Introduction to Complex Reflection Groups and Their Braid Groups

 Springer

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# Preface

Weyl groups are finite groups acting as reflection groups on rational vector spaces. It is well known that these rational reflection groups appear as “skeletons” of many important mathematical objects: algebraic groups, Hecke algebras, Artin–Tits braid groups, etc.

By extension of the base field, Weyl groups may be viewed as particular cases of finite *complex reflection groups*, *i.e.*, finite subgroups of some  $\mathrm{GL}_r(\mathbb{C})$  generated by (pseudo-)reflections. Such groups have been characterized by Shephard–Todd and Chevalley as those finite subgroups of  $\mathrm{GL}_r(\mathbb{C})$  whose ring of invariants in the corresponding symmetric algebra  $\mathbb{C}[X_1, X_2, \dots, X_r]$  is a regular graded ring (a polynomial algebra). The irreducible finite complex reflection groups have been classified by Shephard–Todd.

It has been recently discovered that complex reflection groups play also a key role in the structure as well as in the representation theory of finite reductive groups *i.e.*, rational points of algebraic connected reductive groups over a finite field – for a survey on that type of questions, see for example [Bro1]. Not only do complex reflection groups appear as “automizers” of peculiar tori (the “cyclotomic Sylow subgroups”), but as much as Weyl groups, they give rise to braid groups and generalized Hecke algebras which govern representation theory of finite reductive groups.

In the meantime, it has been understood that many of the known properties of Weyl groups, and more generally of Coxeter finite groups (reflection groups over  $\mathbb{R}$ ) can be generalized to complex reflection groups – although in most cases new methods have to be found. The most spectacular result in that direction, due to Bessis [Bes3], states that the complement of the hyperplanes arrangement of a complex reflection group is  $K(\pi, 1)$ . The oldest (but not least important), due to Steinberg [St], states that the subgroup which fixes a subspace is still a complex reflection group (a “parabolic subgroup”).

Besides, questions coming from Harmonic Analysis have brought interesting new results on the knowledge of complex reflection groups (see for example [Op1]).

The purpose of this set of Notes (which was written while the author was delivering a graduate course at the University of California Berkeley during the Spring of 2008) is to give a somewhat complete treatment

of the foundations and basic properties of complex reflection groups (characterization, Steinberg theorem, Gutkin–Opdam matrices, Solomon theorem and applications, etc.) including the basic results of Springer [Sp] and Springer–Lehrer [LeSp] on eigenspaces of elements of complex reflection groups. On our way, we also introduce basic definitions and properties of the associated braid groups, as well as a quick introduction to Bessis lifting of Springer theory to braid groups.

As a consequence of our point of view – mainly aimed at further applications to braid groups, cyclotomic Hecke algebras, finite reductive groups – our base fields have characteristic zero.

# Acknowledgement

These notes would never have existed without more than a decade of mathematical collaboration with Jean Michel and our frequent (and sometimes hard) discussions. I owe him several of the approaches chosen here, as well as many suggestions. Moreover, the tables reproduced in the appendix are extract from tables he built.

My thanks also go to the students of Berkeley who attended this course until the end and pushed me to be clearer.

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