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Vector Fields on Singular Varieties

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Preface

Vector fields on manifolds play major roles in mathematics and other sciences. In particular, the Poincaré–Hopf index theorem and its geometric counterpart, the Gauss–Bonnet theorem, give rise to the theory of Chern classes, key invariants of manifolds in geometry and topology.

One has often to face problems where the underlying space is no more a manifold but a singular variety. Thus it is natural to ask what is the "good" notion of index of a vector field, and of Chern classes, if the space acquires singularities. The question was explored by several authors with various answers, starting with the pioneering work of M.-H. Schwartz and R. MacPherson.

We present these notions in the framework of the obstruction theory and the Chern–Weil theory. The interplay between these two methods is one of the main features of the monograph.

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We would like to express our indebtedness to M.-H. Schwartz for her great contribution to the theory, that has played and keeps playing an important role in our mathematics.

Contents

Introduction

The study of vector fields and flows near an isolated singularity, or stationary point, has played for decades, and even centuries, a major role in several areas of mathematics and in other sciences, notably in physics, biology, economics, etc. The most basic invariant of a vector field at an isolated singularity is its local Poincaré–Hopf index, which has been studied from very many different viewpoints and there is a vast literature about it. At the same time, it is becoming more and more usual to face problems and situations in Mathematics (and in other sciences) where the underlying space is not a manifold but a singular variety. It is thus natural to ask what should be the "good" notion of index of a vector field on a singular variety, depending on which properties of the local index we have in mind.

For instance one has the theorem of Poincaré–Hopf saying that the sum of the local indices of a vector field with isolated singularities on a closed oriented manifold, is independent of the choice of the vector field and equals the Euler–Poincaré characteristic of the manifold in question. Defining an index for vector fields on singular varieties that has this property leads to the Schwartz index, that we explain below.

Similarly, an important property of the local Poincaré–Hopf index is that it is stable under perturbations, or in other words, that if we approximate the given vector field by another vector field which has only Morse singularities, then the local index of the initial vector field is the number of singularities of its morsification counted with signs. Defining an index for vector fields on singular varieties that has this property leads to a different index, the GSV index.

There are other important properties of the local Poincaré–Hopf index that give rise to various other indices when we look at singular varieties. That makes the study of indices of vector fields over singular varieties an interesting field of current research, which combines an amazing variety of ideas and techniques coming from algebraic topology, differential geometry, algebraic geometry, dynamical systems, mathematical physics, etc.

The goal of this monograph is to give an account of the various indices of vector fields on singular varieties that are in the literature, the relations among them, and the way how they relate with various generalizations of

Chern classes to singular varieties. Indices of vector fields and Chern classes of vector bundles are nowadays present in many branches of mathematics, and these two concepts are linked together in an essential way.

This monograph goes together with [28] to give a global view of the theory of indices and Chern classes for singular spaces. In [28] the focus is on the theory of characteristic classes for singular varieties. Here the emphasis is on indices and their relation with Chern classes. We do this following two of the classical viewpoints for studying Chern classes, both introduced by Chern himself. These are the topological viewpoint, thinking of Chern classes as being the primary obstruction to constructing sections of appropriate fiber bundles, and the differential-geometric viewpoint, via Chern–Weil theory, where the corresponding classes are localized at the "singularities" of certain connections via the theory of residues, which is largely indebted to R. Bott.

The interplay between these two viewpoints for studying indices and characteristic classes, obstruction theory and Chern–Weil theory, is a key feature of this monograph.

This work does not pretend to be comprehensive, and yet it offers a global viewpoint of the theory of indices of vector fields and Chern classes of singular varieties that can be of interest for people working in singularities, algebraic and differential geometry, algebraic topology, and even in string theory and mathematical physics. In each individual chapter we indicate additional references to the literature, for further reading.

The study of indices of vector fields and Chern classes for singular varieties started in the early 1960s with M.-H. Schwartz, and then continued by R. MacPherson and many others. This is today an active field of research, in which the foundations of the theory are being laid out by several authors, and so are their relations with other branches of geometry, topology, and singularity theory.

We start Chap. 1 with the basic, well-known, theory of indices of vector fields and Chern classes that we need in the sequel, and we describe for manifolds the two viewpoints that we use in the rest of the work to study these invariants, namely localization via obstruction theory and localization via Chern–Weil theory.

In Chap. 2 begins the discussion of indices of vector fields on singular varieties. We start with the index introduced by M.-H. Schwartz (in [139,141]) in her study of Chern classes for singular varieties. For her purpose there was no point in considering vector fields in general, but only a special class of vector fields (and frames) that she called "radial," which are obtained by the important process of *radial extension* that she introduced. The generalization of this index to other vector fields was first done by H. King and D. Trotman in [96], and later independently in [6,49,149]. We call this the *Schwartz index*; in the literature it is also called "radial index" because it measures how far the vector field is from being radial. The corresponding discussion for frames is done in Chap. 10.

As mentioned above, one of the basic properties of the local index of Poincaré–Hopf is that it is stable under perturbations. If we now consider an analytic variety V defined, say, by a holomorphic function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with an isolated critical point at 0, and if v is a vector field on V, nonsingular away from 0, then one would like "the index" of v at 0 to be stable under small perturbations of both, the function f and the vector field v. This leads naturally to another concept of index, now called the *GSV index,* introduced in [71, 144, 149], and this is the topic we envisage in Chap. 3.

This monograph mostly concerns vector fields on complex analytic varieties; nevertheless, it is of course interesting to consider also the real analytic case. This is what we do in Chap. 4, where we present the work of M. Aguilar, J. Seade and A. Verjovsky in [6]. That chapter begins with a definition of the Schwartz index in this setting, done independently in [6, 49, 96]. Then we discuss the GSV index, which is now an integer if the singular variety V is odd-dimensional, and an integer modulo 2 if the dimension of V is even. The information one gets is related to previous work by M. Kervaire, C.T.C. Wall and others, and provides an extension of the concept of Milnor number for real analytic map-germs with isolated singularities which may not be algebraically isolated. The viewpoint considered in this chapter is topological; indices of vector fields on real analytic varieties are also considered in Chap. 7 from an algebraic viewpoint, following the work of X. Gómez-Mont et al.

Chapter 5 concerns the virtual index, introduced in [111] by D. Lehmann, M. Soares and T. Suwa for holomorphic vector fields on complex analytic varieties; the extension to continuous vector fields was done in [31, 149]. This index is defined via Chern–Weil theory. The idea comes from the fact that a vector field with an isolated singularity provides a localization of the top dimensional Chern class at the singular point of the vector field, and the number one gets is the corresponding local index of Poincaré–Hopf. Similarly, if $(V, 0)$ is an isolated complete intersection singularity germ in $\mathbb{C}^{n+k}$, an ICIS for short, defined by functions $f = (f_1, \ldots, f_k)$, then a tangent vector field on V, with an isolated singularity, together with the gradient vector fields of the f_i, defines a localization at 0 of the nth Chern class of the ambient space, and the number one gets is *the virtual index* of the vector field. In this context the virtual index coincides with the GSV index, however the definition of the virtual index actually extends to the general setting of vector fields with compact singular set defined on complex analytic varieties which are "strong" local complete intersections.

The previous indices are all defined for continuous vector fields on singular varieties. However, in many situations the vector fields in question are actually analytic, and this is the setting we envisage in Chaps. 6 and 7.

If the vector field is holomorphic, the localization theory via Chern–Weil becomes richer because of the Bott vanishing theorem, producing further interesting residues; this is the topic we study in Chap. 6. A holomorphic vector field defines by integration a one-dimensional holomorphic foliation, and the theory of residues can be developed for general singular holomor-

phic foliations on certain singular varieties. We consider here one dimensional singular holomorphic foliations, and we refer to [156] for a systematical treatment of the general case. We have three types of residues that arise from a Bott type vanishing theorem: (i) generalizations of Baum–Bott residues to singular varieties, first introduced in [13, 14]; (ii) the Camacho–Sad index, introduced in [42] and used to prove the separatrix theorem. Nowadays there are many generalizations of this index; (iii) variations, introduced in [93] and generalized in [113] (see also [39, 40]). For a local separatrix of a holomorphic vector field, the variation equals the sum of the GSV and the Camacho–Sad indices (see Chap. 6 or Proposition 5 in [40]).

Another remarkable property of the index of Poincaré–Hopf is that, in the case of a germ of holomorphic vector field $v = \sum_{i=1}^{n} h_i \frac{\partial}{\partial z_i}$ on $\mathbb{C}^n$ with an isolated singularity at 0, the local index equals the integer:

$$\dim_{\mathbb{C}} \mathcal{O}_n \big/ (h_1, \ldots, h_n)\,,$$

where $(h_1, \ldots, h_n)$ is the ideal generated by the components of v in the ring $\mathcal{O}_n$ of germs of holomorphic functions at 0 in $\mathbb{C}^n$. This and other facts motivated the search for algebraic formulas for the index of vector fields on singular varieties. The *homological index* of X. Gómez-Mont [68] is a answer to that search. It considers an isolated singularity germ $(V, 0)$ of arbitrary dimension, and a holomorphic vector field on V, singular only at 0. One has the Kähler differentials on V, and a Koszul complex $(\Omega^{\bullet}_{V,0}, v)$:

$$0 \longrightarrow \Omega^{n}_{V,0} \longrightarrow \Omega^{n-1}_{V,0} \longrightarrow \cdots \longrightarrow \mathcal{O}_{V,0} \longrightarrow 0\,,$$

where the arrows are given by contracting forms by the vector field v. The homological index of v is defined to be the Euler characteristic of this complex. If the ambient space V is smooth at 0, the complex is exact in all dimensions, except in degree 0 where the corresponding homology group has dimension equal to the local Poincaré–Hopf index of v at 0. If $(V, 0)$ is an ICIS, the recent article [17] of H.-C. Graf von Bothmer, W. Ebeling and X. Gómez-Mont shows that this index coincides with the GSV index, a fact previously known only for vector fields on hypersurface germs. We remark however that the homological index is defined for vector fields on arbitrary isolated normal singularity germs, while the GSV index is only defined on complete intersection germs. Hence the homological index does provide a new invariant for singular varieties which is not yet understood in general. It would be interesting to know what this index measures globally, *i.e.*, given a compact variety W with isolated singularities and a holomorphic vector field on it with isolated singularities, its total homological index an invariant of W. What type of invariant is it? If W is a local complete intersection, this is just the usual Euler–Poincaré characteristic of a smoothing of W, and as explained in the text, this equals the 0-degree Fulton–Johnson class of W.

In Chap. 7 we briefly describe the homological index, as well as work in this spirit done for real analytic vector fields by X. Gómez-Mont, P. Mardešić and L. Giraldo, generalizing to vector fields on real analytic hypersurface germs the signature formula of Eisenbud–Levin and Khimshiashvili for the local index of real analytic vector fields in $\mathbb{R}^n$.

Chapter 8 concerns the local Euler obstruction, introduced by R. MacPherson in [117] for constructing Chern classes of complex varieties. In [33], J.-P. Brasselet and M.-H. Schwartz defined this invariant via vector fields, interpretation that was essential to prove (also in [33]) that the Schwartz classes of singular varieties coincide with MacPherson's classes via Alexander duality. This viewpoint brings the local Euler obstruction into the framework of "indices of vector fields on singular varieties" and yields to another index, that we may call *the local Euler obstruction* of Whitney stratified vector fields with isolated singularities; the classical local Euler obstruction corresponding to the case of the radial vector field. The Brasselet–Schwartz "Proportionality Theorem" of [33] shows that this index plays an important role when considering liftings of stratified vector fields to sections of the Nash bundle. If the vector field in question comes from the gradient of a function on the singular variety, this local Euler obstruction is the "defect" studied in [32]. By [150], this invariant measures the number of critical points of a local perturbation of the given function which are contained in the regular part of the singular variety, and it is related to several generalizations of the Milnor number to the case of functions on singular varieties.

When considering smooth (real) manifolds, the tangent and cotangent bundles are canonically isomorphic and it does not make much difference to consider either vector fields or 1-forms in order to define their indices and their relations with Chern classes. If the ambient space is a complex manifold, this is no longer the case, but there are still ways for comparing indices of vector fields and 1-forms, and to use these to study Chern classes of manifolds. To some extent this is also true for singular varieties, but there are however important differences and each of the two settings has its own advantages.

R. MacPherson defined the notion of local Euler obstruction in terms of indices of 1-forms on singular varieties. Such indices also appear in the work of C. Sabbah [134,135], particularly in relation with the local Euler obstruction. The systematic study of indices of 1-forms on singular varieties started in a series of articles by W. Ebeling and S. Gusein-Zade This has been, to some extent, a study parallel to the one for vector fields, outlined above. This is the subject of Chap. 9, briefly discussed in this monograph for completeness.

The last part of this work, Chaps. 10–13, concerns several generalizations of Chern classes to the case of singular varieties from the viewpoint of localization theory, by means of indices of vector fields. We refer to [28] for a detailed account on characteristic classes for singular varieties from a global point of view.

In his 1946 original paper [44], S. S. Chern gave several equivalent definitions of his classes, with diverse points of view. In the case of singular

varieties, there are several definitions of characteristic classes, given by various authors. They correspond to various extensions one has of the concept of "tangent bundle" as we go from manifolds to singular varieties. Each of these viewpoints leads to a generalization of Chern classes to the case of singular varieties, as described in [28]. In this monograph we focus on the relations of Chern classes with various indices of vector fields and frames, considering the following four generalizations of the tangent bundle:

(i) the union of the spaces tangent to each stratum of a Whitney stratification of the singular variety;

(ii) the Nash bundle over the Nash blow up of the singular variety;

(iii) the virtual bundle $TM|_V - N|_V$ if the variety V is defined by a holomorphic section of some bundle N over a complex manifold M;

(iv) the tangent sheaf over the singular variety V.

The first generalization is due to M.-H. Schwartz in [139, 141], considering a singular complex analytic variety V embedded in a smooth one M which is equipped with a Whitney stratification adapted to V; she considers a class of stratified frames to define characteristic classes of V which do not depend on M nor on the various choices. These classes live in the cohomology of M with support in V, *i.e.*, $H^*(M, M \setminus V)$. Alexander duality takes this cohomology into the homology of V, and if V is nonsingular the classes one gets in $H_*(V)$ are the homology Chern classes of the manifold, *i.e.*, the Poincaré duals of the usual Chern classes. The 0-dimensional part is the Euler–Poincaré characteristic, which can be localized at the singular set of a vector field and the local contribution is the Schwartz index. The generalization of this index for frames and its relation with Chern classes are given in Chap. 10.

The second extension of the concept of tangent bundle, given by the Nash bundle $\widetilde{T} \to \widetilde{V}$ over the Nash transform $\widetilde{V}$, was used by R. MacPherson [117] to define Chern classes for singular varieties. First one gets the Mather classes of V, also introduced in [117], which are by definition the image of the Chern classes of $\widetilde{T}$ carried into its homology via the Alexander morphism (which is not an isomorphism in general) and then mapped to the homology of V by the morphism $\nu : H_*(\widetilde{V}) \to H_*(V)$. MacPherson's Chern classes for singular varieties [117] live in the homology of V and can be thought of as being the Mather classes of V weighted by the local Euler obstruction in a sense that is made precise in 10.6. MacPherson's classes satisfy important axioms and functoriality properties conjectured by P. Deligne and A. Grothendieck in the early 1970s.

Later, J.-P. Brasselet and M. H. Schwartz proved in [33] that the Alexander isomorphism $H^*(M, M \setminus V) \cong H_*(V)$ carries the Schwartz classes into MacPherson's classes, so they are now called the *Schwartz–MacPherson classes* of V.

The third way of extending the concept of tangent bundle to singular varieties that we mentioned above was introduced by W. Fulton and K. Johnson in [60]. Notice that if a variety $V \subset M$ is defined by a regular section s of a holomorphic bundle E over M, then the bundle $N = E|_V$ is, on the regular

part V_{reg}, isomorphic to the normal bundle. One has an isomorphism (as C^∞ vector bundles)

$$TM|_{V_{\mathrm{reg}}} = TV_{\mathrm{reg}} \oplus N|_{V_{\mathrm{reg}}} \,,$$

and therefore the virtual bundle $\tau_V = [TM|_V - N|_V]$, regarded as an element in the K-theory group $KU(V)$, is called the virtual tangent bundle of V. The homology Chern classes of the virtual tangent bundle τ_V are the *Fulton–Johnson classes* of V. In this book we envisage only the case, where V is a local complete intersection in the complex manifold M. When localized at the singular set of a vector field, the local contribution to the 0-dimensional part of the Fulton–Johnson class is the virtual index. When V has only isolated singularities, this corresponds to the Euler–Poincaré characteristic of a smoothing of V. The generalization of the virtual index for frames and its relation with Chern classes are given in Chap. 11.

In general, these classes are different from the Schwartz–MacPherson classes. If V has only isolated singularities, then (by [149,155]) the Schwartz–MacPherson and Fulton–Johnson classes coincide in all dimensions other than 0, and in dimension 0 this difference is given by the local Milnor numbers of V at its singular points. Hence it is natural to call *Milnor classes* the difference between Fulton–Johnson and Schwartz–MacPherson classes. These classes were studied by P. Aluffi [8], who called them μ-classes; there have been significant contributions to the subject afterwards, either by Aluffi himself and by various other authors, such as S. Yokura, A. Parusiński and P. Pragacz, D. Lehmann, T. Ohmoto, J. Schürmann, and the authors of this monograph. This is studied in Chap. 12.

Of course one may also compare Chern–Mather with Fulton–Johnson classes. This was done in [125] for (strong) local complete intersections with isolated singularities, using results of [149,155]. As in the previous case, these classes coincide in all dimensions greater than 0; in dimension 0 their difference is given by the polar multiplicities of T. Gaffney. The corresponding study for varieties with nonisolated singularities has not been done yet.

Finally, the fourth way for extending the concept of tangent bundle to singular varieties by considering the tangent sheaf Θ_V, which is by definition the dual of Ω_V, the sheaf of Kähler differentials on V, fits within the framework considered in [158] of Chern classes for coherent sheaves. We briefly describe some of their properties in Chap. 13. In particular, if V is a local complete intersection in M, then one has a canonical locally free resolution of Ω_V and the corresponding Chern classes essentially coincide with the Fulton–Johnson classes, though the corresponding classes for Θ_V differ from these.

In the sequel we explain how the various indices of vector fields that we discuss in Chaps. 2–8 are related among themselves and how they relate to some generalization of the Chern classes of manifolds to the case of singular varieties. There is however something missing in this picture: so far we do not know of a direct relation between the homological index and some type of Chern classes for singular varieties, neither we know of a direct relation between the Chern classes of the tangent sheaf (or its dual) and some index of vector fields (or 1-forms).

While writing this monograph we have tried to convey the reader a unified view of the various generalizations for singular varieties one has of the important concepts of the local index of Poincaré–Hopf and Chern classes of manifolds. These are topics of current research which keep developing and the literature is vast, so we focused on the most classical approaches for this subject. There are of course important topics that were just glanced here, or maybe even not discussed at all, specially concerning new trends in algebraic geometry and topology, such as string theory and motivic integration. Yet, we think the content of this monograph contributes to lay down the foundations of a theory for singular varieties which is just beginning to be developed and understood. This ought to play in the future such an important role for understanding the geometry and topology of singular varieties as they do for manifolds. And this should also have important applications to other branches of knowledge, where it is important to consider vector fields and flows on orbifolds and singular varieties.