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Partial Inner Product Spaces

Theory and Applications

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Foreword

This volume has its origin in a longterm collaboration between Alex Grossmann (Marseille) and one of us (JPA), going back to 1967. This has resulted in a whole collection of notes, manuscripts, and joint papers. In particular, a large set of unpublished notes by AG (dubbed the 'skeleton') has proven extremely valuable for writing the book, and we thank him warmly for putting it at our disposal. JPA also thanks the Centre de Physique Théorique (CPT, Marseille) for its hearty hospitality at the time.

Later on, almost thirty years ago, the two authors of this book started to interact (with a strong initial impulse of G. Epifanio, CT's advisor at the time), mostly in the domain of partial operator algebras. This last collaboration has consisted entirely of bilateral visits between Louvain-la-Neuve and Palermo. We thank our home institutions for a constantly warm hospitality, as well as various funding agencies that made it possible, namely, the Commisariat Général aux Relations Internationales de la Communauté Française de Belgique (Belgium), the Direzione Generale per le Relazioni Culturali del Ministero degli Affari Esteri Italiano and the Ministero dell'Università e della Ricerca Scientifica (Italy). In the meantime, we also enjoyed the collaboration of many colleagues and students such as F. (Debacker)-Mathot, J-R. Fontaine, J. Shabani (LLN), G. Epifanio, F. Bagarello, A. Russo, F. Tschinke (Palermo), G. Lassner†, K-D. Kürsten (Leipzig), W. Karwowski (Wrocław), and A. Inoue (Fukuoka). We thank them all.

Last, but not least, we owe much to our respective wives Nicole and Adriana for their loving support and patience throughout this work.

Contents

Prologue

In the course of their curriculum, physics and mathematics students are usually taught the basics of Hilbert space, including operators of various types. The justification of this choice is twofold. On the mathematical side, Hilbert space is the example of an infinite dimensional topological vector space that more closely resembles the familiar Euclidean space and thus it offers the student a smooth introduction into functional analysis. On the physics side, the fact is simply that Hilbert space is the daily language of quantum theory, so that mastering it is an essential tool for the quantum physicist.

Beyond Hilbert Space

However, after a few years of practice, the former student will discover that the tool in question is actually insufficient. If he is a mathematician, he will notice, for instance, that Fourier transform is more naturally formulated in the space L^1 of integrable functions than in the space L^2 of square integrable functions, since the latter requires a nontrivial limiting procedure. Thus enter Banach spaces. More striking, a close look at most partial differential equations of interest for applications reveals that the interesting solutions are seldom smooth or square integrable. Physically meaningful events correspond to changes of regime, which mean discontinuities and/or distributions. Shock waves are a typical example. Actually this state of affairs was recognized long ago by authors like Leray or Sobolev, whence they introduced the notion of *weak solution*. Thus it is no coincidence that many textbooks on PDEs begin with a thorough study of distribution theory. Famous examples are those of Hörmander [Hör63] or Lions-Magenes [LM68].

As for physics, it is true that the very first mathematically precise formulation of quantum mechanics is that of J. von Neumann [vNe55], in 1933, which by the way yielded also the first exact definition of Hilbert space as we know it. However, a pure Hilbert space formulation of quantum mechanics is both inconvenient and foreign to the daily behavior of most physicists, who stick to the more suggestive version of Dirac [Dir30]. A glance at the

textbook of Prugovečki [Pru71] will easily convince the reader.... An additional drawback is the universal character of Hilbert space: all separable Hilbert spaces are isomorphic, but physical systems are not! It would be more logical that the structure of the state space carry some information about the system it describes. In addition, there are many interesting objects that do not find their place in Hilbert space. Plane waves or δ-functions do not belong in L^2, yet they are immensely useful. The same is true of wave functions belonging to the continuous spectrum of the Hamiltonian.

As a matter of fact, all these objects can receive a precise mathematical meaning as distributions or generalized functions, that is, linear functionals over a space of nice test functions. Thus the door opens on Quantum Mechanics beyond Hilbert space [14]. Many different structures have emerged along this line, such the rigged Hilbert spaces (RHS) of Gel'fand *et al.* [GV64], the equipped Hilbert spaces of Berezanskii [Ber68], the extended Hilbert spaces of Prugovečki [170], the analyticity/trajectory spaces of van Eijndhoven and de Graaf [Eij83, EG85, EG86] or the nested Hilbert spaces (NHS) of Grossmann [114]. Among these, the RHS is the best known and it answers the objections made above to the sole use of Hilbert space. A different approach to its introduction is via the consideration of unbounded operators representing observables, as proposed independently by J. Roberts [171, 172], A. Böhm [47], and one of us (JPA) [8, 9]. We will discuss this approach at length in Chapter 5.

The central topic of this volume, namely *partial inner product spaces (*PIP*-spaces)*, has its origin in the first meeting between A. Grossmann and JPA, in 1967. Both of us were already working beyond Hilbert space, with NHS for AG and RHS for JPA. We realized that we were in fact basically doing the same thing, using different languages. After many discussions, we were able to extract the quintessence of our respective approaches, namely, the notions of *partial inner product* and *partial inner product space* (PIP-space). A thorough analysis followed, that led to a number of joint publications [12, 13, 17–19], later with W. Karwowski [22, 23]. Students joined in, such as F. Mathot [Mat75], A-M. Nachin [Nac72] and J. Shabani [177]. But gradually interest moved to other subjects, such as algebras of unbounded operators and partial operator algebras, culminating in the monograph by the two of us with A. Inoue [AIT02]. But sometimes PIP-spaces came back on the stage also when considering partial *-algebras. Indeed, in their study on partial *-algebras of distribution kernels, Epifanio and Trapani [77] introduced the notion of *multiplication framework*, to be developed later by Trapani and Tschinke [183] when analysing the multiplication of operators acting in a RHS. A multiplication framework is nothing but a family of intermediate spaces (interspaces) between the smallest space and the largest one of a RHS and these spaces indeed generate a true PIP-space.

But, on the whole, the topic of PIP-spaces remained dormant for a number of years, until one of us (JPA) was drawn back into it by the mathematical considerations of the signal processing community. There, indeed, it is

commonplace to exploit families of function or distribution spaces that are indexed by one or several parameters controlling, for instance, regularity or behavior at infinity. Such are the Lebesgue spaces $\{L^p, 1 \leqslant p \leqslant \infty\}$, the Wiener amalgam spaces $W(L^p, \ell^q), 1 \leqslant p, q \leqslant \infty$, the modulation spaces $M_m^{p,q}, 1 \leqslant p, q \leqslant \infty$, the Besov spaces $B_{pq}^s, 1 \leqslant p, q \leqslant \infty$. The interesting point is that individual spaces have little individual value, it is the whole family that counts. Taking into account the duality properties among the various spaces, one concludes that, in all such cases, the underlying structure is that of a PIP-space. In addition, one needs operators that are defined over all spaces of the family, such as translation, modulation or Fourier transform. And the PIP-space formalism yields precisely such a notion of global operator.

Thus it seemed to us that time was ripe for having a second look at the subject and write a synthesis, the result being the present volume.

About the Contents of the Book

The work is organized as follows. We begin by a short introductive chapter, in which we restrict ourselves to the simplest case of a chain or a lattice of Hilbert spaces or Banach spaces. This allows one to get a feeling about the general theory and, in particular, about the machinery of operators on such spaces. The following two chapters are the core of the general theory. It is convenient to divide our study of PIP-spaces into two stages.

In Chapter 1 we consider only the algebraic aspects, focusing of the generation of a PIP-spaces from a so-called *linear compatibility relation* on a vector space V and a partial inner product defined exactly on compatible pairs of vectors. Standard examples are the space ω of *all* complex sequences, with the partial inner product inherited from ℓ^2, whenever defined, and the space $L^1_{\rm loc}(X, d\mu)$ of all measurable, locally integrable, functions on a measure space (X, μ), with the partial inner product inherited from L^2. The key notion here is that of *assaying subspaces*, particular subspaces of V which are in a sense the building blocks of the whole construction. Given a linear compatibility # on a vector space V, it turns out that the set of all assaying subspaces, (partially) ordered by inclusion, is a complete involutive lattice denoted by $\mathcal{F}(V, \#)$. This will lead us to another equivalent formulation, in terms of particular coverings of V by families of subspaces. Now the complete lattice $\mathcal{F}(V, \#)$ defined by a given linear compatibility can be recovered from much smaller families of subspaces, called *generating families.* An interesting observation is that, in many cases, including the two standard examples mentioned above, there exists a generating family consisting entirely of Hilbert spaces. The existence of such generating families in crucial for practical applications; indeed they play the same role as a basis of neighborhoods or a basis of open sets does in topology. And in fact, they will naturally lead to the introduction, in Chapter 2, of a reduced structure called an *indexed* PIP*-space.*

We conclude the chapter with the problem of comparing different compatibilities on the same vector space. *A priori* several order relations may be considered. It turns out that the useful definition is to say that a given compatibility $\#_1$ on V is *coarser* than another one $\#_2$ if, and only if, the complete lattice $\mathcal{F}(V, \#_1)$ is a sublattice of $\mathcal{F}(V, \#_2)$, on which the two involutions coincide (*involutive* sublattice). This concept is useful for the construction of PIP-space structures on a given vector space V. Most vector spaces used in mathematical physics carry a natural (partial) inner product, defined on a suitable domain $\Gamma \subseteq V \times V$. With trivial restrictions on Γ (symmetry, bilinearity), the condition: $f \# g \Leftrightarrow \{f, g\} \in \Gamma$, actually defines a linear compatibility $\#$ on V. Then *all* linear compatibilities which are admissible for that particular inner product are precisely those that are coarser than $\#$, which in turn are determined by all involutive sublattices of $\mathcal{F}(V, \#)$. On the other hand, the problem of refining a given compatibility (and then a given PIP-space structure) admits in general no solution, even less a unique maximal one. However, partial answers to the refinement problem can be given, but some additional structure is needed, namely topological restrictions on individual assaying subsets. This will be the main topic of Chapter 5.

Then, in Chapter 2, we introduce topologies on the assaying subspaces. With a basic nondegeneracy assumption, the latter come as compatible pairs $(V_r, V_{\overline{r}})$, which are dual pairs in the sense of topological vector spaces. This allows one to consider various canonical topologies on these subspaces and explore the consequences of their choice. It turns out that the structure so obtained is extremely rich, but may contain plenty of pathologies. Since the goal of the whole construction is to provide an elementary substitute to the theory of distributions, we are led to consider a particular case, in which all assaying subspaces are of the same type, Hilbert spaces or reflexive Banach spaces. The resulting structure is called an *indexed* PIP*-space*, of type (H), resp. type (B). However, a further restriction is necessary. Indeed, in such a case, the two spaces of a dual pair $(V_r, V_{\overline{r}})$ are conjugate duals of each other, but we require now, in addition, that each of them is given with an explicit norm, not only a normed topology, and the two norms are supposed to be conjugate to each other also. In that case, we speak of a *lattice of Hilbert spaces (LHS)*, resp. a *lattice of Banach spaces (LBS)*. These are finally the structures that are useful in practice, and plenty of examples will be described in the subsequent chapters.

Chapter 3 is devoted to the other central topic of the book, namely, operators on (indexed) PIP-spaces. As we have seen so far, the basic idea of PIP-spaces is that vectors should not be considered individually, but only in terms of the assaying subspaces V_r, which are the basic units of the structure. Correspondingly, an operator on a PIP-space should be defined in terms of assaying subspaces only, with the proviso that only continuous or bounded operators are allowed. Thus an operator is what we will call a *coherent collection* of continuous operators. Its domain is a nonempty union of assaying subspaces of V and its restriction to each of these is linear and bounded into

the target space. In addition, and this is the crucial condition, the operator is *maximal*, in the sense that it has no proper extension satisfying the two conditions above. Requiring this essentially eliminates all the pathologies associated to unbounded operators and their extensions, while at the same time allowing more singular objects.

Once the general definition of operator on a PIP-space is settled, we may turn to various classes, that more or less mimick the standard notions. For instance, regular and totally regular operators, homomorphisms and isomorphisms, unitary operators (with application to group representations), symmetric operators. The last class examplifies what we said above, for it leads to powerful generalizations of various self-adjointness criteria for Hilbert space operators, even for very singular ones (the central topic here is the well-known KLMN theorem). For instance, this technique allows one to treat correctly very singular Schrödinger operators (Hamiltonians with various δ-potentials). In the last section, we turn to another central object, that of a projection operator, and the attending notion of subspace. It turns out that an appropriate definition of PIP-subspace permits to reproduce the familiar Hilbert space situation, namely the bijection between projections and closed subspaces. In addition, this leads to interesting results about finite dimensional subspaces and pre-Hilbert spaces.

Chapter 4 is a collection of concrete examples of PIP-spaces. There are two main classes, spaces of (locally) integrable functions and spaces of sequences. The simplest example of the former is the family of Lebesgue space L^p, $1 < p < \infty$, first over a finite interval (in which case, one gets a chain of Banach spaces), then over $\mathbb{R}$ or $\mathbb{R}^n$, where a genuine lattice is generated. A further generalization is the (wide) class of Köthe function spaces, which contains, among others, most of the spaces of interest in signal processing (see Chapter 8). The other class consists of the Köthe sequence spaces, which incidentally provide most of the pathological situations about topological vector spaces! Next we briefly describe the so-called analyticity/trajectory spaces, which were actually meant as a substitute to distribution theory, better adapted to a rigorous formulation of Dirac's formalism of quantum mechanics. Another class of PIP-spaces concludes the chapter, namely spaces of analytic functions. Starting from the familiar Bargmann space of entire functions [42, 43], we consider first a LBS that generates it (also defined by Bargmann). Then we turn to spaces of functions analytic in a sector. The PIP-space structure we describe, inspired by the work of van Winter in quantum scattering theory [188, 189], leads to a new insight into the latter. In the same way, we present some PIP-space variations around the Bergman or Hardy spaces of functions analytic in a disk.

In Chapter 5, we return to the problem of refinement of PIP-space structures, in particular, the extension from a discrete chain to a continuous one, and similarly for a lattice. When the individual spaces are Banach spaces, we are clearly in the realm of interpolation theory. In the Hilbert case, one can also exploit the spectral theorem of self-adjoint operators. The simplest

example is that of the canonical chain generated by the powers of a positive self-adjoint operator in a Hilbert space, where both techniques can be used. The next case is that of a genuine lattice of Hilbert spaces. In both cases, there are infinitely many solutions. Next we explore how a RHS $\mathcal{D} \subset \mathcal{H} \subset \mathcal{D}^\times$ can be refined into a LHS (here $\mathcal{H}$ is a Hilbert space, $\mathcal{D}$ a dense subspace, endowed with a suitable finer topology, and $\mathcal{D}^\times$ its strong conjugate dual). This is an old problem, connected to the proper definition of a multiplication rule for operators on a RHS. The key is the introduction of the so-called *interspaces*, that is, subspaces $\mathcal{E}$ such that $\mathcal{D} \subset \mathcal{E} \subset \mathcal{D}^\times$. This is, of course, strongly reminiscent of Laurent Schwartz's hilbertian subspaces of a topological vector space and corresponding kernels [175]. Indeed, the crucial condition on interspaces may already be found in that paper. As an application, we construct a family of Banach spaces that generalize the well-known Bessel potential or Sobolev spaces. We also discuss the PIP-space structure of distribution spaces. In particular, we review the elegant construction of the so-called Hilbert spaces of type S of Grossmann, which enables one to construct manageable spaces of nontempered distributions.

The next step is the construction of PIP-spaces generated by a family or an algebra of unbounded operators, equivalently, a (compatible) family of quadratic forms on a Hilbert space. In particular, if one starts from the algebra of regular operators on a PIP-space V, one ends up with *two* PIP-space structures on the same vector space. Comparison between the two leads to several situations, from the 'natural' one to a downright pathological one. Examples may be given for all cases, and this might give some hints for a classification of PIP-spaces.

In Chapter 6, we consider the set $\mathrm{Op}(V)$ of all operators on a PIP-space V as a partial *-algebra. This concept, developed at length in the monograph by Antoine-Inoue-Trapani [AIT02], sheds new light on the operators. Of particular interest is the case where the PIP-space V is a RHS. The proper definition of a multiplication scheme in that context has generated some controversies in the literature [135, 136], but the PIP-space point of view eliminates the pathologies unearthed in these papers. In the same vein, we consider also the construction of representations of partial *-algebras, in particular, the Gel'fand–Naĭmark–Segal (GNS) construction suitably generalized to the PIP-space context. As for general partial *-algebras, one has to take account of the fact that the product of two operators is not always defined, which requires replacing positive linear functionals by sesquilinear ones, in particular the so-called *weights*. Clearly this kind of topic implies borrowing ideas and techniques from operator algebras.

The last two chapters are devoted to applications of PIP-spaces. In Chapter 7, we consider applications in mathematical physics, in the next one applications in signal processing. We begin with quantum mechanics. As mentioned at the beginning of this prologue, the insufficient character of a pure Hilbert space formulation led mathematical physicists to introduce the RHS approach, which then generalizes in straightforward way to a PIP-space

approach, via the consideration of the observables characterizing a physical system (the so-called *labeled* observables). A different generalization that we quickly mention is that of the analyticity/trajectory spaces. A spectacular application of the RHS point of view is a mathematically correct treatment of very singular interactions (δ-potentials or worse). A case where a PIP-space formulation yields a new insight is that of quantum scattering theory, along the lines developed by van Winter [188, 189]. At play here are the spaces of functions analytic in a sector, described in Chapter 4. Also we obtain a precise link to the dilation analyticity or complex scaling method (CSM), nowadays a workhorse in quantum chemistry. We also make some remarks on the still controversial time-asymmetric quantum mechanics, which is based on an energy-valued RHS. The next topic where PIP-spaces are used since a long time is, of course, quantum field theory. In the axiomatic Wightman formulation, based on (tempered) distributions, a RHS language emerges naturally. Two explicit instances, that we describe in some detail, are the construction of the theory from the so-called Borchers algebra and the Euclidean approach of Nelson. Similarly, a proper definition of *unsmeared* field operators require some sort of PIP-space structure. Another area where PIP-spaces have been exploited is that of Lie group representations, using Nelson's theory of analytic vectors, that we touch briefly for concluding the chapter.

The final Chapter 8 is devoted to applications in signal processing. Namely, we explore in some detail a number of families of function spaces that yield the 'natural' framework for some specific applications. Typically, each class is indexed by two indices, at least. One of them characterizes the local behavior (local growth, smoothness), whereas the other specifies the global behavior, for instance the decay properties at infinity. The first example is that of mixed-norm Lebesgue spaces and Wiener amalgam spaces (the first spaces of this type were introduced by N. Wiener in his study of Tauberian theorems). For instance, the amalgam space $W(L^p, \ell^q)$ consists of functions on $\mathbb{R}$ which are locally in L^p and such that the L^p norms over the intervals $(n, n+1)$ form an ℓ^q sequence. This clearly corresponds to the local vs. global behavior announced above. It turns out that such spaces (and generalizations thereof) provide a natural framework for the time-frequency analysis of signals. The same may be said, *a fortiori*, for the modulation spaces $M^{p,q}_m$, which are defined in terms of the Short-Time Fourier (or Gabor) Transform (m is a weight function and $1 \leqslant p, q \leqslant \infty$). Among these, a special role is played by the space $M^{1,1}_1$, called the Feichtinger algebra and denoted usually by $\mathcal{S}_0$ (it is indeed an algebra both under pointwise multiplication and under convolution). $\mathcal{S}_0$ is a reflexive Banach space and one has indeed $\mathcal{S} \subset \mathcal{S}_0 \subset L^2 \subset \mathcal{S}_0^\times \subset \mathcal{S}^\times$ (thus $\mathcal{S}_0$ and its conjugate dual are interspaces in the Schwartz RHS). In practice, $\mathcal{S}_0$ may often advantageously replace Schwartz's space $\mathcal{S}$, yielding the prototypical Banach Gel'fand triple $\mathcal{S}_0 \subset L^2 \subset \mathcal{S}_0^\times$, which plays an important role in time-frequency analysis.

A second important class is that of Besov spaces, which are intrinsically related to the (discrete) wavelet transform. Typical results concern the specification of a space to which a given function belongs through the decay properties of its wavelet coefficients in an appropriate wavelet basis. Finally, we survey briefly a far reaching generalization of all the preceding spaces, namely, the so-called co-orbit spaces. These spaces are defined in terms of an integrable representation of a suitable Lie group. For instance, the Weyl-Heisenberg group leads to modulation spaces, the affine group of the line yields Besov spaces, SL(2,$\mathbb{R}$) gives Bergman spaces.

For the convenience of the reader, we conclude the volume with two short appendices. The first one (A) gives some indications about the so-called Galois connections (used in Chapter 1), and the second (B) collects some basic facts about (locally convex) topological vector spaces, mostly needed in Chapter 2.

A final word about the presentation. Although a large literature already exists on the subject, we have decided to mention very few papers in the body of the chapters. Instead, each of them concludes with notes that give all the relevant bibliography. We have tried, in particular, to trace most of the results to the original papers. Thus a substantial part of the book consists of a survey of known results, often reformulated in the PIP-space language. This means that, in most cases, we state and comment the relevant results, but skip the proofs, referring instead to the literature. Clearly there are omissions and misrepresentations, due to our own ignorance and prejudices. We take responsibility for this and apologize in advance to those authors whose work we might have mistreated.

Jean-Pierre Antoine (Louvain-la-Neuve)
Camillo Trapani (Palermo)

Introduction: Lattices of Hilbert or Banach Spaces and Operators on Them

I.1 Motivation

It is a fact that many function spaces that play a central role in analysis come in the form of families, indexed by one or several parameters that characterize the behavior of functions (smoothness, behavior at infinity, ...). The simplest structure is a *chain of Hilbert or (reflexive) Banach spaces.* Let us give two familiar examples.

(i) *The Lebesgue L^p spaces on a finite interval*, e.g. $\mathcal{I} = \{L^p([0,1], dx), 1 \leqslant p \leqslant \infty\}$:

$$L^\infty \subset \ldots \subset L^{\overline{q}} \subset L^{\overline{r}} \subset \ldots \subset L^2 \subset \ldots \subset L^r \subset L^q \subset \ldots \subset L^1, \tag{I.1}$$

where $1 < q < r < 2$. Here L^q and $L^{\overline{q}}$ are dual to each other $(1/q + 1/\overline{q} = 1)$, and similarly $L^r, L^{\overline{r}}$ $(1/r + 1/\overline{r} = 1)$. By the Hölder inequality, the (L^2) inner product

$$\langle f|g\rangle = \int_0^1 \overline{f(x)}\, g(x)\, dx \tag{I.2}$$

is well-defined if $f \in L^q$, $g \in L^{\overline{q}}$. However, it is *not* well-defined for two arbitrary functions $f, g \in L^1$. Take for instance, $f(x) = g(x) = x^{-1/2}$: $f \in L^1$, but $fg = f^2 \notin L^1$. Thus, on L^1, (I.2) defines only a *partial* inner product. The same result holds for any finite interval of $\mathbb{R}$ instead of [0,1].

(ii) *The chain of Hilbert spaces built on the powers of a positive self-adjoint operator* $A \geqslant 1$ in a Hilbert space $\mathcal{H}_0$. Let $\mathcal{H}_n$ be $D(A^n)$, the domain of A^n, equipped with the graph norm $\|f\|_n = \|A^n f\|$, $f \in D(A^n)$, for $n \in \mathbb{N}$ or $n \in \mathbb{R}^+$, and $\mathcal{H}_{\overline{n}} := \mathcal{H}_{-n} = \mathcal{H}_n^\times$ (conjugate dual):

$$\mathcal{D}^\infty(A) := \bigcap_n \mathcal{H}_n \subset \ldots \subset \mathcal{H}_2 \subset \mathcal{H}_1 \subset \mathcal{H}_0 \subset \mathcal{H}_{\overline{1}} \subset \mathcal{H}_{\overline{2}} \ldots \subset \mathcal{D}_{\overline{\infty}}(A) := \bigcup_n \mathcal{H}_n. \tag{I.3}$$

Note that here the index n may be integer or real, the link between the two cases being established by the spectral theorem for self-adjoint operators.

Here again the inner product of $\mathcal{H}_0$ extends to each pair $\mathcal{H}_n, \mathcal{H}_{\overline{n}}$, but on $\mathcal{D}_{\overline{\infty}}(A)$ it yields only a *partial* inner product. The following examples, all three in $\mathcal{H}_0 = L^2(\mathbb{R}, dx)$ are standard:

- $(A_{\mathrm{p}} f)(x) = (1 + x^2)^{1/2} f(x)$.
- $(A_{\mathrm{m}} f)(x) = (1 - \frac{d^2}{dx^2})^{1/2} f(x)$.
- $(A_{\mathrm{osc}} f)(x) = (1 + x^2 - \frac{d^2}{dx^2}) f(x)$.

(The notation is suggested by the operators of position, momentum and harmonic oscillator energy in quantum mechanics, respectively). In the case of A_{m}, the intermediate spaces are the Bessel potential (or Sobolev) spaces $H^s(\mathbb{R})$, $s \in \mathbb{Z}$ or $\mathbb{R}$. Note that both $\mathcal{D}^\infty(A_{\mathrm{p}}) \cap \mathcal{D}^\infty(A_{\mathrm{m}})$ and $\mathcal{D}^\infty(A_{\mathrm{osc}})$ coincide with the Schwartz space $\mathcal{S}(\mathbb{R})$ of smooth functions of fast decay, and $\mathcal{D}_{\overline{\infty}}(A_{\mathrm{osc}})$ with the space $\mathcal{S}^\times(\mathbb{R})$ of tempered distributions.[1]

However, a moment's reflection shows that the total order relation inherent in a chain is in fact an unnecessary restriction, partially ordered structures are sufficient, and indeed necessary in practice. For instance, in order to get a better control on the behavior of individual functions, one may consider the lattice built on the powers of A_{p} and A_{m} simultaneously. Then the extreme spaces are still $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}^\times(\mathbb{R})$. Similarly, in the case of several variables, controlling the behavior of a function in each variable separately requires a nonordered set of spaces. This is in fact a statement about tensor products (remember that $L^2(X \times Y) \simeq L^2(X) \otimes L^2(Y)$). Indeed a glance at the work of Palais on chains of Hilbert spaces shows that the tensor product of two chains of Hilbert spaces, $\{\mathcal{H}_n\} \otimes \{\mathcal{K}_m\}$ is naturally a lattice $\{\mathcal{H}_n \otimes \mathcal{K}_m\}$ of Hilbert spaces. For instance, in the example above, for two variables x, y, that would mean considering intermediate Hilbert spaces corresponding to the product of two operators, $\big(A_{\mathrm{m}}(x)\big)^n \big(A_{\mathrm{m}}(y)\big)^m$.

Thus the structure we want to analyze is that of *lattices of Hilbert or Banach spaces*. Many examples are around us, for instance the lattice generated by the spaces $L^p(\mathbb{R}, dx)$, the amalgam spaces $W(L^p, \ell^q)$, the mixed norm spaces $L^{p,q}_m(\mathbb{R}, dx)$, and many more (these spaces will be discussed in detail in Chapters 4 and 8, where the references to original papers will be given). In all these cases, which contain most families of function spaces of interest in analysis and in signal processing, a common structure emerges for the "large" space V, defined as the union of all individual spaces. There is a lattice of Hilbert or reflexive Banach spaces V_r, with an (order-reversing) involution $V_r \leftrightarrow V_{\overline{r}}$, where $V_{\overline{r}} = V_r^\times$ (the space of continuous antilinear functionals on V_r), a central Hilbert space $V_o \simeq V_{\overline{o}}$, and a partial inner product on V that extends the inner product of V_o to pairs of dual spaces $V_r, V_{\overline{r}}$.

Actually, in many cases, it is the family $\{V_r\}$ as a whole that is meaningful, not the individual spaces. The spaces $L^p(\mathbb{R})$ are a good example. Therefore,

[1] considered here as continuous *conjugate linear* functionals on $\mathcal{S}$. See the Notes to Chapter 1, Section 1.1.

many operators should be considered globally, for the whole chain or lattice, instead of on individual spaces. For instance, in many spaces of interest in signal processing, this would apply to operators implementing translations ($x \mapsto x-y$) or dilations ($x \mapsto x/a$), convolution operators, Fourier transform, etc. In the same spirit, it is often useful to have a *common* basis for the whole family of spaces, such as the Haar basis for the spaces $L^p(\mathbb{R})$, $1 < p < \infty$. Thus we need a notion of operator and basis defined globally for the chain or lattice itself.

The subject matter of the present volume is to present a formalism that answers these questions, namely, the theory of *partial inner product spaces* or PIP-*spaces*. However, before analyzing in detail the general theory, we will concentrate in this introductory chapter on the simple case of lattices of Hilbert or Banach spaces and operators on them. These are indeed the most useful families of spaces for the applications.

I.2 Lattices of Hilbert or Banach Spaces

I.2.1 Definitions

Let thus $\mathcal{J} = \{\mathcal{H}_p,\, p \in J\}$ be a family of Hilbert spaces, partially ordered by inclusion (the index set J has the same order structure). Then $\mathcal{J}$ generates a lattice $\mathcal{I}$, indexed by I, by the operations:

- $\mathcal{H}_{p\wedge q} = \mathcal{H}_p \cap \mathcal{H}_q$, with the projective norm

$$\|f\|^2_{p\wedge q} = \|f\|^2_p + \|f\|^2_q, \;; \tag{I.4}$$

- $\mathcal{H}_{p\vee q} = \mathcal{H}_p + \mathcal{H}_q$, the vector sum, with the inductive norm

$$\|f\|^2_{p\vee q} = \inf_{f=g+h} \left(\|g\|^2_p + \|h\|^2_q\right),\; g \in \mathcal{H}_p, f \in \mathcal{H}_q\,. \tag{I.5}$$

It turns out that both $\mathcal{H}_{p\wedge q}$ and $\mathcal{H}_{p\vee q}$ are Hilbert spaces, that is, they are complete with the norms indicated. These statements will be proved, and the corresponding mathematical structure analyzed, in Chapter 2, Section 2.2.

Assume that the original index set J has an involution $q \leftrightarrow \overline{q}$, with $\mathcal{H}_{\overline{q}} = \mathcal{H}_q^\times$ (by an involution, we mean a one-to-one correspondence such that $p \leqslant q$ implies $\overline{q} \leqslant \overline{p}$ and $\overline{\overline{p}} = p$). Then the lattice $\mathcal{I}$ inherits the same duality structure, with $\mathcal{H}_{p\wedge q} \leftrightarrow \mathcal{H}_{\overline{p}\vee\overline{q}}$ (it is then called an *involutive lattice*; a precise definition will be given in Section 1.1). Finally, we assume the family $\mathcal{J}$ contains a unique self-dual space $V_o = V_{\overline{o}}$. The resulting structure is called a *lattice of Hilbert spaces* or LHS.

In addition to the family $\mathcal{I} = \{V_r,\, r \in I\}$, it is convenient to consider the two spaces $V^\#$ and V defined as

$$V = \sum_{q \in I} \mathcal{H}_q, \quad V^{\#} = \bigcap_{q \in I} \mathcal{H}_q. \tag{I.6}$$

These two spaces themselves usually do *not* belong to $\mathcal{I}$.

The concept of LHS is closely related to that of nested Hilbert space, which will be discussed in Section 2.4). More important, this construction is the basic structure of interpolation theory.

A similar construction can be performed with a family $\mathcal{J} = \{V_p,\, p \in J\}$ of reflexive Banach spaces, the resulting structure being then a *lattice of Banach spaces* or LBS. In this case, one considers the following norms, which are usual in interpolation theory.

- $V_{p\wedge q} = V_p \cap V_q$, with the *projective* norm

$$\|f\|_{p\wedge q} = \|f\|_p + \|f\|_q\,; \tag{I.7}$$

- $V_{p\vee q} = V_p + V_q$, with the *inductive* norm

$$\|f\|_{p\vee q} = \inf_{f=g+h} \left(\|g\|_p + \|h\|_q\right), g \in V_p, f \in V_q. \tag{I.8}$$

Here too, we assume the family $\mathcal{J}$ contains a unique self-dual space $V_o = V_{\overline{o}}$, which is a Hilbert space.

I.2.2 Partial Inner Product on a LHS/LBS

The basic question is how to generate such structures in a systematic fashion. In order to answer it, we may reformulate it as follows: given a vector space V and two vectors $f, g \in V$, when does their inner product make sense? A way of formalizing the answer is given by the idea of *compatibility*.

Let $\mathcal{I} := \{V_r,\, r \in I\}$ be a LHS or a LBS and $f, g \in V$ two vectors. Then we say that f and g are *compatible*, which we note $f\#g$, if the following relation holds:

$$f\#g \;\Leftrightarrow\; \exists\, r \in I \text{ such that } f \in V_r, g \in V_{\overline{r}}\,. \tag{I.9}$$

Clearly the relation # is a symmetric binary relation which preserves linearity:

$$\begin{aligned} f\#g &\iff g\#f,\ \forall\, f, g \in V, \\ f\#g, f\#h &\implies f\#(\alpha g + \beta h),\ \forall\, f, g, h \in V,\ \forall\, \alpha, \beta \in \mathbb{C}. \end{aligned}$$

From now on, we write $\mathcal{I} = (V, \#)$. A formal definition will be given in Section 1.1.

Now we introduce the basic notions of our structure, namely, a partial inner product and a partial inner product space.

A *partial inner product* on $(V, \#)$ is a hermitian form $\langle\cdot|\cdot\rangle$ defined exactly on compatible pairs of vectors, that is, on $\Delta = \left(\bigcup_{V_r\in\mathcal{I}} V_r \times V_{\overline{r}}\right) \bigcup \left(V^{\#} \times V\right)$. A *partial inner product space (PIP-space)* is a vector space V equipped with a linear compatibility and a partial inner product.

In general, the partial inner product is not required to be positive definite, but it will be in all the examples given in this chapter. In the present case of a LHS or a LBS $\mathcal{I} = \{V_r, r \in I\}$, this simply means that the partial inner product is defined between elements of two spaces in duality $(V_r, V_{\overline{r}})$; in particular, for $r = o$, it is a Hermitian form on the Hilbert space V_o, which we take as equal to the inner product. Thus, the partial inner product may be seen as the extension of the inner product of V_o to the whole of V, whenever possible. Clearly, with the compatibility (I.9) and this definition of partial inner product, the LHS/LBS $\mathcal{I}$ is a PIP-space, that we henceforth denote as $(V, \#, \langle\cdot|\cdot\rangle)$.

The partial inner product defines a notion of *orthogonality* : $f \perp g$ if and only if $f\#g$ and $\langle f|g\rangle = 0$. Then we say that the PIP-space $(V, \#, \langle\cdot|\cdot\rangle)$ is *nondegenerate* if $(V^{\#})^{\perp} = \{0\}$, that is, if $\langle f|g\rangle = 0$ for all $f \in V^{\#}$ implies $g = 0$.

In this introductory chapter, we will assume that our PIP-space $(V, \#, \langle\cdot|\cdot\rangle)$ is nondegenerate. This assumption has important topological consequences, that will be explored at length in Chapter 2. In a nutshell, $(V^{\#}, V)$, like every couple $(V_r, V_{\overline{r}})$, $r \in I$, is a dual pair in the sense of topological vector spaces. Furthermore, $r < s$ implies $V_r \subset V_s$, and the embedding operator $E_{sr} : V_r \to V_s$ is continuous and has dense range. In particular, $V^{\#}$ is dense in every V_r.

I.2.3 Two Examples of LHS

Let us give two simple examples of LHS, thus of PIP-spaces as well.

(i) Sequence spaces

Let V be the space ω of *all* complex sequences $x = (x_n)$. Consider the following family of weighted Hilbert spaces, which are obviously subspaces of ω:

$$\ell^2(r) = \{(x_n) \in \omega : (x_n/r_n) \in \ell^2, i.e., \sum_{n=1}^{\infty} |x_n|^2\, r_n^{-2} < \infty\}, \tag{I.10}$$

where $r = (r_n)$, $r_n > 0$, is a sequence of positive numbers. The family possesses an involution:

$$\ell^2(r) \leftrightarrow \ell^2(\overline{r}) = \ell^2(r)^{\times}, \text{ where } \overline{r}_n = 1/r_n.$$

In addition, there is a central, self-dual Hilbert space, namely, $\ell^2(1) = \ell^2(\overline{1}) = \ell^2$, where 1 denotes the unit sequence, $r_n = 1$, for all n.

As a matter of fact, the collection $\mathcal{I} := \{\ell^2(r)\}$ of those vector subspaces of ω is an involutive lattice.

(i) $\mathcal{I}$ is a lattice for the following operations:

$$\begin{aligned} \ell^2(r) \wedge \ell^2(s) &= \ell^2(u), \quad \text{where} \;\; u_n = \min\{r_n, s_n\}, \\ \ell^2(r) \vee \ell^2(s) &= \ell^2(v), \quad \text{where} \;\; v_n = \max\{r_n, s_n\}. \end{aligned}$$

Indeed one shows easily that the norms of $\ell^2(u)$ and $\ell^2(v)$ are equivalent, respectively, to the projective and inductive norms defined in (I.4), (I.5) above (proofs will be given in Section 4.3).

(ii) $\mathcal{I}$ is an involutive lattice, with the involution $r \leftrightarrow \overline{r} \equiv (r_n^{-1})$. Indeed:

$$\begin{aligned} [\ell^2(u)]^{\#} &= \ell^2(\overline{u}) = \ell^2(\overline{r}) \vee \ell^2(\overline{s}), \\ [\ell^2(v)]^{\#} &= \ell^2(\overline{v}) = \ell^2(\overline{r}) \wedge \ell^2(\overline{s}). \end{aligned}$$

Actually, $\mathcal{I}$ is a sublattice of $\mathcal{L}(\omega)$, the lattice of all vector subspaces of ω, i.e.,

$$\begin{aligned} \ell^2(r) \wedge \ell^2(s) &= \ell^2(r) \cap \ell^2(s), \\ \ell^2(r) \vee \ell^2(s) &= \ell^2(r) + \ell^2(s). \end{aligned}$$

As for the extreme spaces, it is easy to see that the family $\{\ell^2(r)\}$ generates the space ω of *all* complex sequences, while the intersection is the space φ of all *finite* sequences:

$$\bigcup_{r \in I} \ell^2(r) = \omega, \qquad \bigcap_{r \in I} \ell^2(r) = \varphi.$$

Thus, with the partial inner product $\langle x|y\rangle = \sum_{n=1}^{\infty} \overline{x_n}\, y_n$, inherited from $V_o = \ell^2$, the family $\mathcal{I} = \{\ell^2(r)\}$ is a nondegenerate LHS.

(ii) Spaces of locally integrable functions

Instead of sequences, we consider locally integrable functions, i.e., Lebesgue measurable functions, integrable over compact subsets, $f \in L^1_{\rm loc}(\mathbb{R}, dx)$, and define again weighted Hilbert spaces:

$$L^2(r) = \{f \in L^1_{\rm loc}(\mathbb{R}, dx) : f r^{-1} \in L^2, i.e. \int_{\mathbb{R}} |f(x)|^2\, r(x)^{-2}\, dx < \infty\}, \quad \text{(I.11)}$$

with $r, r^{-1} \in L^2_{\rm loc}(\mathbb{R}, dx), r(x) > 0$ a.e. The family $\mathcal{I} = \{L^2(r)\}$ has an involution, $L^2(r) \leftrightarrow L^2(\overline{r})$, with $\overline{r} = r^{-1}$, and a central, self-dual Hilbert space, $L^2(\mathbb{R}, dx)$. This is, of course, the continuous analogue of the preceding

example. Thus we get exactly the same structure as in (i), namely the family $\mathcal{I} = \{L^2(r)\}$ is an involutive lattice, for the operations:

- infimum: $L^2(p \wedge q) = L^2(p) \wedge L^2(q) = L^2(r),\ r(x) = \min(p(x), q(x))$;
- supremum: $L^2(p \vee q) = L^2(p) \vee L^2(q) = L^2(s),\ s(x) = \max(p(x), q(x))$;
- duality: $L^2(p \wedge q) \leftrightarrow L^2(\overline{p} \vee \overline{q}),\ L^2(p \vee q) \leftrightarrow L^2(\overline{p} \wedge \overline{q})$.

Here too, it is easily shown that the lattice $\mathcal{I}$ generates the extreme spaces:

$$\bigcup_{r \in I} L^2(r) = L^1_{\mathrm{loc}}(\mathbb{R}, dx), \qquad \bigcap_{r \in I} L^2(r) = L^\infty_{\mathrm{c}}(\mathbb{R}),$$

where $L^\infty_{\mathrm{c}}(\mathbb{R})$ denotes the space of essentially bounded measurable functions of compact support. With the partial inner product inherited from the central space L^2,

$$\langle f|g\rangle = \int_{\mathbb{R}} \overline{f(x)}g(x)\, dx,$$

the family $\mathcal{I} = \{L^2(r)\}$ becomes a nondegenerate LHS. The construction extends trivially to $\mathbb{R}^n$, or to any manifold (X, μ). It may also be done around Fock space, instead of L^2 (see Section 1.1.3, Example (iv)).

I.3 Operators on a LHS/LBS

As follows from the compatibility relation (I.9), the basic idea of LHS/LBS (and, more generally, PIP-spaces, as we shall see in the next chapter) is that vectors should not be considered individually, but only in terms of the subspaces V_r $(r \in I)$, the building blocks of the structure. Correspondingly, an operator on such a space should be defined in terms of the defining subspaces only, with the proviso that only *bounded* operators between Hilbert or Banach spaces are allowed. Thus an operator is a *coherent collection* of bounded operators. More precisely,

Definition I.3.1. Given a LHS or LBS $V_I = \{V_r, r \in I\}$, an *operator* on V_I is a map A from a subset $\mathcal{D} \subseteq V$ into V, where

(i) $\mathcal{D}$ is a nonempty union of defining subspaces of V_I;
(ii) for every defining subspace V_q contained in $\mathcal{D}$, there exists a $p \in I$ such that the restriction of A to V_q is linear and continuous into V_p (we denote this restriction by A_{pq});
(iii) A has no proper extension satisfying (i) and (ii), i.e., it is maximal.

According to Condition (iii), the domain $\mathcal{D}$ is called the *natural domain* of A and denoted $\mathcal{D}(A)$.

The linear bounded operator $A_{pq} : V_q \to V_p$ is called a *representative* of A. In terms of the latter, the operator A may be characterized by the set $\mathsf{j}(A) = \{(q,p) \in I \times I : A_{pq} \text{ exists}\}$. Thus the operator A may be identified with the collection of its representatives,

$$A \simeq \{A_{pq} : V_q \to V_p : (q,p) \in \mathsf{j}(A)\}.$$

We also need the two sets obtained by projecting $\mathsf{j}(A)$ on the "coordinate" axes, namely,

$$\mathsf{d}(A) = \{q \in I : \text{there is a } p \text{ such that } A_{pq} \text{ exists}\},$$
$$\mathsf{i}(A) = \{p \in I : \text{there is a } q \text{ such that } A_{pq} \text{ exists}\}.$$

The following properties are immediate:

- $\mathsf{d}(A)$ is an initial subset of I: if $q \in \mathsf{d}(A)$ and $q' < q$, then $q' \in \mathsf{d}(A)$, and $A_{pq'} = A_{pq}E_{qq'}$, where $E_{qq'}$ is a representative of the unit operator (this is what we mean by a 'coherent' collection).
- $\mathsf{i}(A)$ is a final subset of I: if $p \in \mathsf{i}(A)$ and $p' > p$, then $p' \in \mathsf{i}(A)$ and $A_{p'q} = E_{p'p}A_{pq}$.
- $\mathsf{j}(A) \subset \mathsf{d}(A) \times \mathsf{i}(A)$, with strict inclusion in general.

We denote by $\mathrm{Op}(V_I)$ the set of all operators on V_I. Of course, a similar definition may be given for operators $A : V_I \to Y_K$ between two LHSs or LBSs.

Since $V^{\#}$ is dense in V_r, for every $r \in I$, an operator may be identified with a sesquilinear form on $V^{\#} \times V^{\#}$. Indeed, the restriction of any representative A_{pq} to $V^{\#} \times V^{\#}$ is such a form, and all these restrictions coincide (these sesquilinear forms are even separately continuous for appropriate topologies on $V^{\#}$, see Chapter 3). Equivalently, an operator may be identified with a linear map from $V^{\#}$ into V (here also continuity may be obtained). But the idea behind the notion of operator is to keep also the *algebraic operations* on operators, namely:

(i) *Adjoint* $A^{\times}$*:* every $A \in \mathrm{Op}(V_I)$ has a unique adjoint $A^{\times} \in \mathrm{Op}(V_I)$, defined by the relation

$$\langle A^{\times}x|y\rangle = \langle x|Ay\rangle, \text{ for } y \in V_r,\ r \in \mathsf{d}(A), \text{ and } x \in V_{\overline{s}},\ s \in \mathsf{i}(A),$$

that is, $(A^{\times})_{\overline{rs}} = (A_{sr})^*$ (usual Hilbert/Banach space adjoint).

It follows that $A^{\times\times} = A$, for every $A \in \mathrm{Op}(V_I)$: no extension is allowed, by the maximality condition (iii) of Definition I.3.1.

(ii) *Partial multiplication:* AB is defined if and only if there is a $q \in \mathsf{i}(B) \cap \mathsf{d}(A)$, that is, if and only if there is a continuous factorization through some V_q :

$$V_r \xrightarrow{B} V_q \xrightarrow{A} V_s, \quad \text{i.e.,} \quad (AB)_{sr} = A_{sq}B_{qr}\,.$$

It is worth noting that, for a LHS/LBS, the natural domain $\mathcal{D}(A)$ is always a vector subspace of V (this is not true for a general PIP-space). Therefore, $\mathrm{Op}(V_I)$ is a vector space and a *partial *-algebra*.

The concept of PIP-space operator is very simple, yet it is a far reaching generalization of bounded operators. It allows indeed to treat on the same footing all kinds of operators, from bounded ones to very singular ones. By this, we mean the following, loosely speaking. Given $A \in Op(V_I)$, when looked at from the central Hilbert space $V_o = \mathcal{H}$, there are three possibilities:

- if $(o, o) \in \mathsf{j}(A)$, i.e., A_{oo} exists, then A corresponds to a bounded operator $V_o \to V_o$;
- if $(o, o) \notin \mathsf{j}(A)$, but there is an $r < o$ such that $(r, o) \in \mathsf{j}(A)$, i.e., A_{oo} does not exist, but only $A_{or} : V_r \to V_o$, with $r < o$, then A corresponds to an unbounded operator A_{or}, with hilbertian domain containing V_r;
- if $(r, o) \notin \mathsf{j}(A)$, for any $r \leqslant o$, i.e., no A_{or} exists, then A is a sesquilinear form on some V_s, $s \leqslant o$, and, as an operator on $\mathcal{H}$, its domain does not contain any V_r (it may be reduced to $\{0\}$): then A corresponds to a singular operator; this happens, for instance, if $(r, s) \in \mathsf{j}(A)$ with $r < o < s$, i.e., there exists $A_{sr} : V_r \to V_s$.

Exactly as for Hilbert or Banach spaces, one may define various types of operators between PIP-spaces, in particular LBS/LHS, such that regular operators, orthogonal projections, homomorphisms and isomorphisms, symmetric operators, unitary operators, etc. We will describe those classes in detail in Chapter 3.

In the following chapters, we will extend this discussion to a general PIP-space and operators between two PIP-spaces. A slightly more restrictive structure, called an *indexed* PIP*-space*, will also be introduced. Many concrete examples will be discussed in detail.

Notes

Section I.1. For unbounded operators, see Reed-Simon I [RS72, Section VIII. 2]. For the spectral theorem, see Kato [Kat76, RS72] or Reed-Simon I [RS72, Section VIII. 1]. The work of Palais may be found in [162, Chap.XIV].

Section I.2. Nested Hilbert spaces were introduced by Grossmann [114].

- For interpolation theory, one may consult the monographs of Bergh-Löfström [BL76] or Triebel [Tri78a].
- Our standard references for topological vector spaces are the monographs of Köthe [Köt69] and Schaefer [Sch71].

Section I.3. Partial *-algebras are studied in detail in the monograph of Antoine-Inoue-Trapani [AIT02].