

Lecture Notes in Mathematics

1985

Editors:

J.-M. Morel, Cachan

F. Takens, Groningen

B. Teissier, Paris

Torsten Linß

Layer-Adapted Meshes for Reaction-Convection-Diffusion Problems

Torsten Linß
Technical University of Dresden
Institute for Numerical Mathematics
Zellescher Weg 12-14
01062 Dresden
Germany
torsten.linss@tu-dresden.de

ISBN: 978-3-642-05133-3 e-ISBN: 978-3-642-05134-0

DOI: 10.1007/978-3-642-05134-0

Springer Heidelberg Dordrecht London New York

Lecture Notes in Mathematics ISSN print edition: 0075-8434

ISSN electronic edition: 1617-9692

Library of Congress Control Number: 2009940926

Mathematics Subject Classification (2000): 65L11, 65L12, 65L20, 65L50, 65L60, 65N06, 65N08, 65N12, 65N30, 65N50

© Springer-Verlag Berlin Heidelberg 2010

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilm or in any other way, and storage in data banks. Duplication of this publication or parts thereof is permitted only under the provisions of the German Copyright Law of September 9, 1965, in its current version, and permission for use must always be obtained from Springer. Violations are liable to prosecution under the German Copyright Law.

The use of general descriptive names, registered names, trademarks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

Cover design: SPi Publisher Services

Printed on acid-free paper

springer.com

Preface

This is a book on numerical methods for singular perturbation problems – in particular, stationary reaction-convection-diffusion problems exhibiting layer behaviour. More precisely, it is devoted to the construction and analysis of layer-adapted meshes underlying these numerical methods.

Numerical methods for singularly perturbed differential equations have been studied since the early 1970s and the research frontier has been constantly expanding since. A comprehensive exposition of the state of the art in the analysis of numerical methods for singular perturbation problems is [141] which was published in 2008. As that monograph covers a big variety of numerical methods, it only contains a rather short introduction to layer-adapted meshes, while the present book is exclusively dedicated to that subject.

An early important contribution towards the optimisation of numerical methods by means of special meshes was made by N.S. Bakhvalov [18] in 1969. His paper spawned a lively discussion in the literature with a number of further meshes being proposed and applied to various singular perturbation problems. However, in the mid 1980s, this development stalled, but was enlivened again by G.I. Shishkin's proposal of piecewise-equidistant meshes in the early 1990s [121, 150]. Because of their very simple structure, they are often much easier to analyse than other meshes, although they give numerical approximations that are inferior to solutions on competing meshes. Shishkin meshes for numerous problems and numerical methods have been studied since and they are still very much in vogue.

With this contribution we try to counter this development and lay the emphasis on more general meshes that – apart from performing better than piecewise-uniform meshes – provide a deeper insight in the course of their analysis.

In this monograph, a classification and a survey are given of layer-adapted meshes for reaction-convection-diffusion problems. The monograph aims at giving a structured and comprehensive account of current ideas in the numerical analysis for various methods on layer-adapted meshes. Both finite differences, finite elements and finite volumes will be covered.

While for finite difference schemes applied to one-dimensional problems, a rather complete convergence theory for arbitrary meshes is developed, the theory is more fragmentary for other methods and problems. They still require the restriction to certain classes of meshes.

The roots of this monograph are a survey lecture presented at the Oberwolfach seminar *Numerical Methods for Singular Perturbation Problems*, 8–14 April 2001 organised by Pieter W. Hemker, Hans-Görg Roos and Martin Stynes, and a review article [91] invited by Thomas J.R. Hughes. I am indebted to their invitations and their continued encouragement.

My thanks also go to a series of colleagues I had the pleasure of working with over the years and who consequently influenced this monograph: Sebastian Franz, Anja Fröhner, R. Bruce Kellogg, Natalia Kopteva, Niall Madden, Hans-Görg Roos, Martin Stynes and Relja Vulcanović.

The finishing work on this monograph was supported by the Science Foundation Ireland during a visit to the University of Limerick and by the Czech Academy of Science through a visiting scholarship.

Dresden and Prague
July 2009

Torsten Linß

Contents

1	Introduction	1
2	Layer-Adapted Meshes	5
2.1	Convection-Diffusion Problems	6
2.1.1	Bakhvalov Meshes	7
2.1.2	Shishkin Meshes	9
2.1.3	Shishkin-Type Meshes	10
2.1.4	Turning-Point Boundary Layers	15
2.1.5	Interior Layers	16
2.1.6	Overlapping Layers	17
2.2	Reaction-Convection-Diffusion Problems	19
2.2.1	Interior Layers	21
2.2.2	Overlapping Layers	23
2.3	Two-Dimensional Problems	24
2.3.1	Reaction-Diffusion Problems	25
2.3.2	Convection-Diffusion	26

Part I One Dimensional Problems

3	The Analytical Behaviour of Solutions	33
3.1	Preliminaries	34
3.1.1	Stability of Differential Operators	34
3.1.2	Green's Functions	36
3.1.3	M -Matrices	37
3.2	Reaction-Convection-Diffusion Problems	38
3.2.1	Stability and Green's Function Estimates	39
3.2.2	Derivative Bounds and Solution Decomposition	45
3.3	Reaction-Diffusion Problems	48
3.3.1	Scalar Reaction-Diffusion Problems	48
3.3.2	Systems of Reaction-Diffusion Equations	52
3.4	Convection-Diffusion Problems with Regular Layers	57
3.4.1	Scalar Convection-Diffusion Problems	57
3.4.2	Weakly Coupled Systems	64
3.4.3	Strongly Coupled Systems	66

3.5	Convection-Diffusion Problems with Turning-Point Layers	69
3.5.1	Stability	69
3.5.2	Derivative Bounds and Solution Decomposition	71
4	Finite Difference Schemes for Convection-Diffusion Problems	77
4.1	Notation	77
4.2	A Simple Upwind Difference Scheme	79
4.2.1	Stability of the Discrete Operator	80
4.2.2	A Priori Error Bounds	84
4.2.3	Error Expansion	87
4.2.4	A Posteriori Error Estimation and Adaptivity	92
4.2.5	An Alternative Convergence Proof	97
4.2.6	The Truncation Error and Barrier Function Technique	100
4.2.7	Discontinuous Coefficients and Point Sources	103
4.2.8	Quasilinear Problems	106
4.2.9	Derivative Approximation	107
4.3	Second-Order Difference Schemes	109
4.3.1	Second-Order Upwind Schemes	109
4.3.2	Central Differencing	119
4.3.3	Convergence Acceleration Techniques	121
4.3.4	A Numerical Example	132
4.4	Systems	134
4.4.1	Weakly Coupled Systems in One Dimension	134
4.4.2	Strongly Coupled Systems	137
4.5	Problems with Turning Point Layers	143
4.5.1	A First-Order Upwind Scheme	144
4.5.2	Convergence on Shishkin Meshes	147
4.5.3	A Numerical Example	148
5	Finite Element and Finite Volume Methods	151
5.1	The Interpolation Error	152
5.2	Linear Galerkin FEM	154
5.2.1	Convergence	154
5.2.2	Supercloseness	156
5.2.3	Gradient Recovery and a Posteriori Error Estimation	160
5.2.4	A Numerical Example	162
5.3	Stabilised FEM	163
5.3.1	Artificial Viscosity Stabilisation	163
5.3.2	Streamline-Diffusion Stabilisation	164

5.4	An Upwind Finite Volume Method	168
5.4.1	Stability of the FVM	171
5.4.2	Convergence in the Energy Norm	175
5.4.3	Convergence in the Maximum Norm.....	180
5.4.4	A Numerical Example	182
6	Discretisations of Reaction-Convection-Diffusion Problems.....	183
6.1	Reaction-Diffusion	183
6.1.1	Linear Finite Elements	184
6.1.2	Central Differencing	190
6.1.3	A Non-Monotone Scheme	202
6.1.4	A Compact Fourth-Order Scheme.....	206
6.2	Systems of Reaction-Diffusion Type	214
6.2.1	The Interpolation Error	214
6.2.2	Linear Finite Elements	215
6.2.3	Central Differencing	217
6.3	Reaction-Convection-Diffusion	221
6.3.1	The Interpolation Error	222
6.3.2	Simple Upwinding	223
 Part II Two Dimensional Problems		
7	The Analytical Behaviour of Solutions	235
7.1	Preliminaries	235
7.1.1	Stability	236
7.1.2	Regularity of Solutions	237
7.2	Reaction-Diffusion	238
7.2.1	Stability	239
7.2.2	Derivative Bounds.....	240
7.3	Convection-Diffusion.....	243
7.3.1	Regular Layers	243
7.3.2	Characteristic Layers.....	245
8	Reaction-Diffusion Problems	247
8.1	Central Differencing	247
8.1.1	Stability	248
8.1.2	Convergence on Layer-Adapted Meshes	249
8.1.3	Numerical Results	253
8.2	Arbitrary Bounded Domains	254
9	Convection-Diffusion Problems	257
9.1	Upwind Difference Schemes	257
9.1.1	Stability	258
9.1.2	Pointwise Error Bounds.....	258
9.1.3	Error Expansion	262

- 9.2 Finite Element Methods 263
 - 9.2.1 The Interpolation Error 264
 - 9.2.2 Galerkin FEM 267
 - 9.2.3 Artificial Viscosity Stabilisation 285
 - 9.2.4 Streamline-Diffusion FEM 289
 - 9.2.5 Characteristic Layers 294
- 9.3 Finite Volume Methods 297
 - 9.3.1 Coercivity of the Method 299
 - 9.3.2 Inverse Monotonicity 302
 - 9.3.3 Convergence 306
- Conclusions and Outlook** 309
- References** 311
- Index** 319

Notation

u	solution of boundary value problems
u^N	numerical approximations of u
$\varepsilon, \varepsilon_d, \varepsilon_c$	perturbation parameter(s)
$\mathcal{L}, \mathcal{L}^*$	differential operator, its adjoint
L, L^*	discrete operator (discretisation of \mathcal{L}), its adjoint
\mathcal{G}, G	continuous and discrete Green's functions
Ω	domain
$\partial\Omega = \Gamma$	boundary of Ω
n	outward pointing unit vector normal to $\partial\Omega$
N	number of mesh intervals (in each coordinate direction)
C	generic constant, independent of ε and N
$\bar{\omega}, \bar{\omega}_x \times \bar{\omega}_y$	sets of mesh points
$h, h_i, \hbar_i, k, k_j, \hbar_j$	mesh step sizes
$v_x, v_{\bar{x}}, v_{\tilde{x}}, v_{\hat{x}}, v_{\check{x}}, v_{\ddot{x}}$	difference operators
u^I	nodal interpolant of u
$\ \cdot\ _\infty$	supremum norm
$\ \cdot\ _1$	L_1 norm
$\ \cdot\ _{-1,\infty}$	$W^{-1,\infty}$ norm
$\ \cdot\ _{\varepsilon,\infty}$	ε -weighted $W^{1,\infty}$ norm
$\ \cdot\ _{*,\omega}$	discrete version of the norm $\ \cdot\ _*$
V^ω	finite element space on the mesh ω
$a(\cdot, \cdot)$	bilinear form
$(\cdot, \cdot), \ \cdot\ _0$	scalar product and norm in $L_2(\Omega)$
$ \cdot _1, \ \cdot\ _\varepsilon$	semi norm and ε -weighted energy norm in $H^1(\Omega)$
$\ \cdot\ _{SD}, \ \cdot\ _\kappa, \ \cdot\ _\rho$	various method-dependent energy norms
$(\cdot, \cdot)_D, \cdot _{*,D}, \ \cdot\ _{*,D}$	scalar product and (semi) norm restricted to $D \subset \Omega$
$C^l, C^{l,\alpha}$	(Hölder) function spaces