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Layer-Adapted Meshes for Reaction-Convection-Diffusion Problems

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Preface

This is a book on numerical methods for singular perturbation problems – in particular, stationary reaction-convection-diffusion problems exhibiting layer behaviour. More precisely, it is devoted to the construction and analysis of layer-adapted meshes underlying these numerical methods.

Numerical methods for singularly perturbed differential equations have been studied since the early 1970s and the research frontier has been constantly expanding since. A comprehensive exposition of the state of the art in the analysis of numerical methods for singular perturbation problems is [141] which was published in 2008. As that monograph covers a big variety of numerical methods, it only contains a rather short introduction to layer-adapted meshes, while the present book is exclusively dedicated to that subject.

An early important contribution towards the optimisation of numerical methods by means of special meshes was made by N.S. Bakhvalov [18] in 1969. His paper spawned a lively discussion in the literature with a number of further meshes being proposed and applied to various singular perturbation problems. However, in the mid 1980s, this development stalled, but was enlivened again by G.I. Shishkin's proposal of piecewise-equidistant meshes in the early 1990s [121, 150]. Because of their very simple structure, they are often much easier to analyse than other meshes, although they give numerical approximations that are inferior to solutions on competing meshes. Shishkin meshes for numerous problems and numerical methods have been studied since and they are still very much in vogue.

With this contribution we try to counter this development and lay the emphasis on more general meshes that – apart from performing better than piecewise-uniform meshes – provide a deeper insight in the course of their analysis.

In this monograph, a classification and a survey are given of layer-adapted meshes for reaction-convection-diffusion problems. The monograph aims at giving a structured and comprehensive account of current ideas in the numerical analysis for various methods on layer-adapted meshes. Both finite differences, finite elements and finite volumes will be covered.

While for finite difference schemes applied to one-dimensional problems, a rather complete convergence theory for arbitrary meshes is developed, the theory is more fragmentary for other methods and problems. They still require the restriction to certain classes of meshes.

The roots of this monograph are a survey lecture presented at the Oberwolfach seminar *Numerical Methods for Singular Perturbation Problems*, 8–14 April 2001 organised by Pieter W. Hemker, Hans-Görg Roos and Martin Stynes, and a review article [91] invited by Thomas J.R. Hughes. I am indebted to their invitations and their continued encouragement.

My thanks also go to a series of colleagues I had the pleasure of working with over the years and who consequently influenced this monograph: Sebastian Franz, Anja Fröhner, R. Bruce Kellogg, Natalia Kopteva, Niall Madden, Hans-Görg Roos, Martin Stynes and Relja Vulanović.

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Notation

u	solution of boundary value problems
u^N	numerical approximations of u
ε, ε_d, ε_c	perturbation parameter(s)
$\mathcal{L}$, $\mathcal{L}^*$	differential operator, its adjoint
L, L^*	discrete operator (discretisation of $\mathcal{L}$), its adjoint
$\mathcal{G}$, G	continuous and discrete Green's functions
Ω	domain
$\partial\Omega = \Gamma$	boundary of Ω
n	outward pointing unit vector normal to $\partial\Omega$
N	number of mesh intervals (in each coordinate direction)
C	generic constant, independent of ε and N
$\bar\omega$, $\bar\omega_x \times \bar\omega_y$	sets of mesh points
h, h_i, $\hbar_i$, k, k_j, $\bar{k}_j$	mesh step sizes
v_x, $v_{\bar x}$, $v_{\check x}$, $v_{\hat x}$, $v_{\bar{x}x}$, $v_{\check{x}x}$	difference operators
u^I	nodal interpolant of u
$\lVert\cdot\rVert_\infty$	supremum norm
$\lVert\cdot\rVert_1$	L_1 norm
$\lVert\cdot\rVert_{-1,\infty}$	$W^{-1,\infty}$ norm
$\lvert\lVert\cdot\rVert\rvert_{\varepsilon,\infty}$	ε-weighted $W^{1,\infty}$ norm
$\lVert\cdot\rVert_{*,\omega}$	discrete version of the norm $\lVert\cdot\rVert_*$
V^ω	finite element space on the mesh ω
$a(\cdot,\cdot)$	bilinear form
$(\cdot,\cdot)$, $\lVert\cdot\rVert_0$	scalar product and norm in $L_2(\Omega)$
$\lvert\cdot\rvert_1$, $\lvert\lVert\cdot\rVert\rvert_\varepsilon$	semi norm and ε-weighted energy norm in $H^1(\Omega)$
$\lvert\lVert\cdot\rVert\rvert_{SD}$, $\lvert\lVert\cdot\rVert\rvert_\kappa$, $\lvert\lVert\cdot\rVert\rvert_\rho$	various method-dependent energy norms
$(\cdot,\cdot)_D$, $\lvert\cdot\rvert_{*,D}$, $\lVert\cdot\rVert_{*,D}$	scalar product and (semi) norm restricted to $D \subset \Omega$
C^l, $C^{l,\alpha}$	(Hölder) function spaces