Lecture Notes in Mathematics 1984

Editors:
J.-M. Morel, Cachan
F. Takens, Groningen
B. Teissier, Paris

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Geometric Description of Images as Topographic Maps

Springer

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ISBN: 978-3-642-04610-0 e-ISBN: 978-3-642-04611-7
DOI: 10.1007/978-3-642-04611-7
Springer Heidelberg Dordrecht London New York

Lecture Notes in Mathematics ISSN print edition: 0075-8434
ISSN electronic edition: 1617-9692

Library of Congress Control Number: 2009938947

Mathematics Subject Classification (2000): 68U10, 94A08, 05C05

Cover design: SPi Publisher Services

Printed on acid-free paper

springer.com

Preface

This book discusses the basic geometric contents of an image and presents a tree data structure to handle it efficiently. It analyzes also some morphological operators that simplify this geometric contents and their implementation in terms of the data structures introduced. It finally reviews several applications to image comparison and registration, to edge and corner computation, and the selection of features associated to a given scale in images.

Let us first say that, to avoid a long list, we shall not give references in this summary; they are obviously contained in this monograph.

A gray level image is usually modeled as a function defined in a bounded domain $D \subseteq \mathbb{R}^N$ (typically $N = 2$ for usual snapshots, $N = 3$ for medical images or movies) with values in $\mathbb{R}$. The sensors of a camera or a CCD array transform the continuum of light energies to a finite interval of values by means of a nonlinear function g. The contrast change g depends on the properties of the sensors, but also on the illumination conditions and the reflection properties of the objects, and those conditions are generally unknown. Images are thus observed modulo an arbitrary and unknown contrast change.

Mathematical morphology recognizes contrast invariance as a basic requirement and proposes that image analysis operators take into account this invariance principle. An image u is thus a representative of an equivalence class of images v obtained from u via a contrast change, i.e., $v = g(u)$ where g, for simplicity, is a continuous strictly increasing function. Under this assumption, the reliable information in the image is contained in the level sets, be they upper $[u \geq \lambda] := \{x \in D : u(x) \geq \lambda\}$, or lower $[u \leq \lambda] := \{x \in D : u(x) \leq \lambda\}$, independently of their actual level. This theory has been described with detail in the books of Jean Serra, and in the forthcoming book of Frédéric Guichard and Jean-Michel Morel. We review the basic ideas at several points in this monograph, in particular, in Chap. 1.

More recently, taking into account local illumination effects, a more local description of the geometric contents of the image was presented. This led to the introduction of connected components of level sets as basic geometric objects of the images. We shall refer to the description of an image in terms of them as a topographic description of the image. The inclusion between upper (resp. lower) level sets gives an obvious tree structure specific to each family

of geometric objects which may be used to encode the image. To be able to handle efficiently both trees in a single data structure which also contains the geometric relations between them, Pascal Monasse was able to fuse both trees into a single one, called the tree of shapes of the image. The basic idea is simple: given a connected component of a level set, be it upper or lower, we fill its holes (an operation that we call saturation). In this way we obtain (in $\mathbb{R}^2$) a simply connected object which can be determined by its boundary and which is called a shape of the image. Since any two shapes of an image are either nested or disjoint, shapes can be structured as a tree, which permits us to handle them in a very efficient way. This is what we call the tree of shapes of an image. We also note that the actual connected components of upper and lower level sets can be recovered from the shapes. To be able to proceed with this construction, we assume that the image is modeled as an upper semicontinuous function. This will be the basic functional model for images adopted in this book. The purpose of Chap. 2 is to introduce the tree of shapes of the image and give its mathematical description. For that we need first to introduce some topological preliminaries and the main properties of the filling saturation operator.

Chapter 3 is devoted to the study of connected operators, which are filters that simplify the topographic map while keeping its essential features. These filters act on the connected components of level sets and simplify the tree of shapes of the images. We shall mainly concentrate on two filters: extrema killers and the grain filter, and we shall study their mathematical properties, and their effect on the tree of shapes of the image. We shall prove that the grain filter is self-dual, i.e., invariant under contrast inversion, and we shall characterize it axiomatically. Let us mention that extrema killers and grain filters can be efficiently implemented using the tree of shapes. We display experiments that illustrate the effect of those filters on images.

The use of a topographic description of images, surfaces, or $3D$ data has been introduced and motivated in different areas of research, including image processing, computer graphics, and geographic information systems (GIS). The motivations for such a description differ depending on the field of application. In all cases these descriptions aim to achieve an efficient description of the basic shapes in the given image and their topological changes as a function of a physical quantity that depends on the type of data (intensity in images, height in data elevation models, etc.).

In computer graphics and geographic information systems, topographic maps represent a high level description of the data. Topographic maps are represented by the contour maps, i.e., the isocontours of the given scalar data. The description of the varying isocontours requires the introduction of data structures, like the *topographic change tree* or *contour tree*, or the Reeb graph, which can represent the nesting of contour lines on a contour map (or a continuous topographic structure). In all cases, the proposed description can be considered as an implementation of Morse theory. Given the scalar data u defined in a domain Ω of $\mathbb{R}^N$ ($u : \Omega \to \mathbb{R}$), the contour map is defined

in the literature as the family of isocontours $[u = \lambda]$, $\lambda \in \mathbb{R}$, or in terms of the boundaries of upper (or lower) level sets $[u \geq \lambda]$ ($[u \leq \lambda]$). The first description is more adapted to the case of smooth data (some interpolation will be required in the case of digital data) while the second description can be adapted to more general continuous data where there are plateaus of constant elevation or discontinuous data.

In the context of computer graphics, Morse theory has also been used to encode surfaces in $3D$ space. Several authors have proposed to use a tree structure like the Reeb graph complemented with information about the Morse indices of the singularities and including enough intermediate contours to be able to reconstruct by interpolation the precise way in which the surface is embedded in $3D$ space.

As we already discussed, in image processing, the topographic description was advocated as a local and contrast invariant description of images (i.e., invariant under illumination changes), and its developments have led to the notion of shape and its efficient description in terms of the tree of shapes of the image. The purpose of Chap. 4 is to give the topological description of the tree of shapes and its singularities. For that we introduce and study a topological weak Morse theory for the tree of shapes of the image, and prove its equivalence to other (topological) weak Morse structures which have been used in the literature (let us mention in particular the work of Kronrod). We describe a simple combinatorial algorithm that gives directly the critical levels of the image. In this chapter images are considered to be continuous functions.

Chapter 5 describes the construction of the tree of shapes of an image by fusion of the trees of connected components of upper and lower level sets. Though this algorithm is less efficient when $N = 2$, where some more effective algorithms exist (they are described in Chaps. 6 and 7), it is still useful when $N \geq 3$.

Chapter 6 is devoted to an algorithm computing the tree of shapes of a digital image. The precise data structures and an efficient implementation of the algorithm is given in details. We also study the theoretical complexity of the algorithm.

Chapter 7 is devoted to the description of the tree of bilinear level lines and its algorithmic implementation. After a bilinear interpolation of the discrete data, the image could be treated as a continuous function and a tree of bilinear level lines $[u = \lambda]$ can be computed. Whereas the algorithm uses the same ideas as the one in Chap. 6, there are meaningful differences, the most prominent of which being that there is an infinite number of bilinear level lines, though their structure is finite in nature, as proved in Chap. 4. The tree of bilinear level lines is related to the contour tree computed with the isocontours of the interpolated image. The work of Kronrod can be considered as a mathematical description of the isocontour tree in the case of two-dimensional functions.

Chapter 8 is devoted to applications. Three main categories will be discussed: image comparison, image registration, and extraction of features like edges, corners, or scale adaptive neighborhoods of each pixel. The problem concerning each application is discussed, and we explain how the data structures developed in the present monograph can be applied for an efficient solution of these problems. The experiments illustrate our claims.

Research on this topic has been initiated by the authors roughly 10 years before publication of this monograph. Theoretical and algorithmic advances, so as diverse applications, have been presented at conferences or published in journals by the authors and other researchers, but the present notes regroup them for the first time in a coherent, unified and self-contained framework. We tried to prove all claims by means of elementary results from topology and analysis. We also included numerous simple figures to illustrate the notions and expose various configurations in proofs. We hope that makes them accessible to the general knowledgeable mathematician. Finally, let us mention that reference implementation of most if not all algorithms are included in the open source software suite MegaWave (`http://megawave.cmla.ens-cachan.fr`).

During the preparation of this text V. Caselles was partially supported by the "ICREA Acadèmia" prize for excellence in research funded by the Generalitat de Catalunya and by PNPGC projects, references BFM2003-02125 and MTM2006-14836. Jean-Michel Morel knows that his constant encouragements were a determinant factor in the completion of these notes and the authors are grateful for his support. They would also like to thank Enric Meinhardt for his contributions to Chap. 5. Finally, the feedback from the anonymous reviewers of this monograph was greatly appreciated.

Barcelona, Paris,
July 2009

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