Lecture Notes in Mathematics 1670

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Sobolev Gradients and Differential Equations

Second Edition

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Preface

What is expected from a theory of differential equations? Look first at the fundamental theorem for ordinary differential equations:

Theorem 0.1. *Suppose that n is a positive integer and G is an open subset of $R \times R^n$ which contains a point (c, w). Suppose also that $f : G \to R^n$ is a continuous function for which there is $M > 0$ such that*

$$\|f(t,x) - f(t,y)\| \leq M\|x - y\| \text{ for all } (t,x),(t,y) \in G. \tag{0.1}$$

Then there is an open interval (a, b) containing c for which there is a unique function u on (a, b) so that

$$u(c) = w,\ u'(t) = f(t, u(t)),\ t \in (a, b).$$

This result can be proved in several constructive ways which yield, along the way, error estimates giving a basis for numerical computation of solutions. Now this existence and uniqueness result certainly does not solve all problems in ordinary differential equations. For one thing, the result is only local. For just one other instance, it doesn't tell about two point boundary value problems, even though it has relevance there. Nevertheless, it provides a position of strength from which to study a wide variety of ordinary differential equations. The fact of existence and uniqueness of a solution gives us something to study in a qualitative, numerical or algebraic setting. The constructive nature of arguments for the above result gives one a good start toward discerning properties of solutions.

Many agree that it would be good to have a similar position of strength for partial differential equations but such does not now exist. It has been argued that there cannot be a central theory of partial differential equations since there is such a great variety of problems. To such an argument I reply that the same opinion about ordinary differential equations was probably held not so much more than a century ago.

These notes are devoted to a description of Sobolev gradients for a variety of problems in differential equations. Sobolev gradients are used in descent processes to find zeros or critical points of functions which in turn provide

solutions to underlying differential equations. Our gradients are generally given constructively and do not require full boundary conditions (*i.e.*, conditions which are necessary and sufficient for existence and uniqueness) to be known beforehand. The processes tend to converge in some (non-Euclidean) sense to a nearest solution. The methods apply in cases which are mixed hyperbolic and elliptic — even cases in which regions of hyperbolicity and ellipticity are determined by nonlinearities. Applications to the problem of transonic flow will illustrate this. Numerics are a natural part of the development given here. In fact, numerics are in a sense ahead of theory, giving a spur to more inquiry.

So, do we arrive at a position of strength for fairly general partial differential equations? Here at least is a shadow of such a theory.

A key thing for a reader to keep in mind is that continuous steepest descent with Sobolev gradients is expressed as an ordinary differential equations in a function space whereas alternative descent methods are often partial differential equations themselves (for example, see Chapter 16 in the case of minimal surface problems).

Notes for Second Edition

The theory of Sobolev gradients has developed a great deal since the publication of the first edition of these notes. Many of these developments are reflected in this second edition, which is about twice the length of the first one.

- The use of Sobolev gradients to find critical points of the Ginzburg-Landau energy functional of superconductivity has greatly expanded. It is now near the design stage for superconducting devices. P. Kazemi's recent discoveries play a substantial role here.
- The treatment of Newton's method in the context of Sobolev gradients has been expanded to include a version of the Nash-Moser inverse function theorem. The problem of 'loss of derivatives' has been avoided entirely, a fact that leads to a relatively simple argument for such inverse function results when applied to differential equations. It was first pointed out by A. Castro that considerations for gradient inequalities have much in common with Moser's development of an inverse function theorem.
- The Tricomi equation, showing both elliptic and hyperbolic regions, has been treated using Sobolev gradients.
- A number of new convergence results for continuous steepest descent are included.
- Work on the hyperbolic Monge-Ampere equation, due to T. Howard, is described. This work opens up a new aspect of the study of such equations.
- Use of Sobolev gradients for nonlinear Schrödinger equations is noted.
- A greatly expanded list of properties of the imbedding operator which connects a Hilbert space with a dense linear subspace which is a Hilbert space in its own right. Much of this is due to P. Kazemi.

- After the first edition of this work was published, it was realized that this author's previous use of what is called 'gradient inequality' was preceded by Łojasiewicz inequalities in finite dimensions.
- There is reference to gradient inequality results work of S. Huang and of R. Chill.
- There is an account of Chan-Hilliard equations by S. Sial, T. Lookman, A. Saxena and the present writer.
- There are Sobolev gradient results for fractal regions.
- Some least squares results are given which have application to the problem of separating actual chaos from apparent chaos induced by discretization.
- A new result is given which relates nonlinear semigroup theory to the problem of boundary or supplementary conditions for partial differential equations.

In the first edition, several authors contributed sections on their work with Sobolev gradients. In the second edition, several have kindly agreed to write a chapter on their work. These include

- A development of numerical integration by means of Sobolev gradients, by Ian Knowles and Robert Wallace.
- A discussion of relationships between Sobolev gradients and preconditioning, by Janos Karatson.
- A presentation of curve fitting in the context of Sobolev gradients, by Robert Renka.
- Results on sign changing solutions and Morse index problems, by John M. Neuberger.
- Oil-water separation, elasticity and Model A problems, by Sultan Sial.

Robert Renka and I have had regular discussions about Sobolev gradients for more than two decades. Many others, particularly John M. Neuberger, have read portions of these notes and have contributed corrections and helpful suggestions. Any remaining errors and obscurities are mine. Many students, colleagues, collaborators and others have provided substantial insights. Any attempt at a list acknowledging this help would contain many names but would likely be inadequate. Hence I have decided to not try to make such a list.

I express profound gratitude to Springer for their help and extraordinary patience.

Contents