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# Stochastic Analysis in Discrete and Continuous Settings

With Normal Martingales

 Springer

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# Preface

This monograph is an introduction to some aspects of stochastic analysis in the framework of normal martingales, in both discrete and continuous time. The text is mostly self-contained, except for Section 5.7 that requires some background in geometry, and should be accessible to graduate students and researchers having already received a basic training in probability. Prerequisites are mostly limited to a knowledge of measure theory and probability, namely  $\sigma$ -algebras, expectations, and conditional expectations. A short introduction to stochastic calculus for continuous and jump processes is given in Chapter 2 using normal martingales, whose predictable quadratic variation is the Lebesgue measure.

There already exists several books devoted to stochastic analysis for continuous diffusion processes on Gaussian and Wiener spaces, cf. e.g. [51], [63], [65], [72], [83], [84], [92], [128], [134], [143], [146], [147]. The particular feature of this text is to simultaneously consider continuous processes and jump processes in the unified framework of normal martingales.

These notes have grown from several versions of graduate courses given in the Master in Imaging and Computation at the University of La Rochelle and in the Master of Mathematics and Applications at the University of Poitiers, as well as from lectures presented at the universities of Ankara, Greifswald, Marne la Vallée, Tunis, and Wuhan, at the invitations of G. Wallet, M. Arnaudon, H. Körezlioğlu, U. Franz, A. Sulem, H. Ouerdiane, and L.M. Wu, respectively. The text has also benefited from constructive remarks from several colleagues and former students, including D. David, A. Joulin, Y.T. Ma, C. Pintoux, and A. Réveillac. I thank in particular J.C. Breton for numerous suggestions and corrections.

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