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# Smooth Ergodic Theory for Endomorphisms

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*Comme souvenir de mon amitié avec le  
Professeur Y. -S. Sun, je voudrais mentionner  
que c'est lui qui m'a fait connaître la formule  
de Pesin pour la première fois de ma vie.*

*Min Qian*

# Preface

Smooth ergodic theory of deterministic dynamical systems deals with the study of dynamical behaviors relevant to certain invariant measures under differentiable mappings or flows. The relevance of invariant measures is that they describe the frequencies of visits for an orbit and hence they give a probabilistic description of the evolution of a dynamical system. The fact that the system is differentiable allows one to use techniques from analysis and geometry.

The study of transformations and their long-term behavior is ubiquitous in mathematics and the sciences. They arise not only in applications to the real world but also to diverse mathematical disciplines, including number theory, Lie groups, algorithms, Riemannian geometry, etc. Hence smooth ergodic theory is the meeting ground of many different ideas in pure and applied mathematics. It has witnessed a great progress since the pioneering works of Sinai, Ruelle and Bowen on Axiom A diffeomorphisms and of Pesin on non-uniformly hyperbolic systems, and now it becomes a well-developed field.

In this theory, among the major concepts are the notions of Lyapunov exponents and metric entropy. Lyapunov exponent describes the exponential rate of expansion or contraction in certain direction along an orbit. Obviously, positive Lyapunov exponents corresponds to the local instability of trajectories. One of the paradigms of dynamical systems is that the local instability of trajectories may lead to the stochastic behavior of the system. Metric entropy, introduced by Kolmogorov and Sinai, is a purely measure-theoretic invariant, which measures the complexity of the dynamical system generated by iterations of the transformation. It has been studied a good deal in abstract ergodic theory, see [75].

The relationship between these two concepts has always been an important problem. A fundamental result concerning this problem is Margulis-Ruelle inequality, which states that the metric entropy can be bounded from above by the sum of positive Lyapunov exponents (See Chapter II). More deep results can be obtained when the system exhibits certain hyperbolicity. The strongest hyperbolicity occurs in the important class of Axiom A systems. In the ergodic theory of Axiom A diffeomorphisms developed by Sinai [88], Ruelle [76] and Bowen [10, 11], it was shown that

for an Axiom A attractor there is a unique invariant measure which is characterized by each of the following properties:

- (1) The metric entropy is equal to the sum of positive Lyapunov exponents.
- (2) The conditional measures of the invariant measure on unstable manifolds are absolutely continuous with respect to the Lebesgue measures on these manifolds.
- (3) Lebesgue almost every point in an open neighborhood of the attractor is generic to this measure.

Property (1) is now known as *Pesin's entropy formula* and property (2) is known as *SRB property* of the invariant measure. Each of these properties has been shown to be significant in its own right, but it is also remarkable that they are equivalent to each other in the case of an Axiom A attractor. In mid-seventies, in a series of papers Pesin developed a machinery to study non-uniformly hyperbolic systems [62, 63]. He obtained a general theorem on the existence and the absolute continuity of invariant families of stable and unstable manifolds of a smooth dynamical system, corresponding to its non-zero Lyapunov exponents. Meanwhile, he also studied the ergodic properties of smooth dynamical systems possessing an absolutely continuous invariant measure. The most striking result is that Pesin's entropy formula also holds in this case. Then it was conjectured by Ruelle and later on proved by Ledrappier, Strelcyn and Young that for an invariant measure of a  $C^2$  diffeomorphism, Pesin's entropy formula holds if and only if it satisfies the SRB property [41, 42]. In other words, the equivalence of properties (1) and (2) can hold in a more general circumstance.

The above results have been successfully generalized to several frameworks. Among them are random iterations of diffeomorphisms and deterministic endomorphisms. For random diffeomorphisms, first initiated by Ledrappier and Young [44], Liu and Qian provided a systematic treatment on the subject [51]. However the results for deterministic system are still scattered in the literature.

The main purpose of this monograph is to summarize these results and to provide a *systematic* treatment on this aspect for *deterministic* systems. The novelty of our treatment lies in the fact that we directly consider endomorphisms throughout the monograph. The results for diffeomorphisms can be obtained as a special case. It is interesting to point out that the method developed to attack *Random Dynamical Systems* [38, 44] can be adapted to treat the endomorphism case. It turns out to be the inverse sequence approach known in the dynamical system theory but it has never been detailed into a systematic treatment as one can see in [44]. Therefore, this monograph gives convincing evidence how deterministic theory can be benefited by probabilistic consideration.

The monograph is organized as follows.

We will review some fundamental concepts in Chapter I. Since the whole monograph mainly deals with endomorphisms with the help of inverse limit space, we also provide the simple relations of entropies and Lyapunov exponents between the base dynamical system and the induced dynamical system on the inverse limit space.

Chapter II is devoted to the Margulis-Ruelle inequality. This inequality was first given by Margulis in the case of diffeomorphisms preserving a smooth measure.

The general statement is due to Ruelle [77]. Rigorous proofs are available in several books for the case of diffeomorphisms only, see [32] and [57]. We present a short and rigorous proof for the general  $C^1$  maps in this chapter.

In Chapter III, we study the simplest case—expanding maps. Although Pesin’s entropy formula is actually a consequence of the main theorem in Chapter VI, there are still some other nice results under weak conditions. This chapter follows from the work of H.-Y. Hu [27].

In Chapter IV, we study the strongest hyperbolic case, the ergodic theory for Axiom A endomorphisms. This chapter is from the work of Qian and Zhang [72].

Chapter V consists of the study of the structure of unstable manifolds. Since in general the unstable manifold at each point depends on the whole backward orbit, for different orbit there might be different unstable manifold at the same point. Therefore, there is no foliation structure of unstable manifolds in these case. We consider the structure of unstable manifolds in the inverse limit space. The source of this chapter is the work of S. Zhu [100] with slight modification.

In Chapter VI we extend Pesin’s entropy formula to the general  $C^2$  endomorphisms. This is done by Liu [46] recently in a different approach.

In Chapter VII we present a formulation of the SRB property for invariant measures of  $C^2$  endomorphisms of a compact manifold via their inverse limit spaces, and then prove that this property is sufficient and necessary for the entropy formula. This is a non-invertible version of the main theorem of [42]. As a nontrivial corollary of this result, an invariant measure of a  $C^2$  endomorphism has this SRB property if it is absolutely continuous with respect to the Lebesgue measure of the manifold. Invariant measures having this SRB property also exist on Axiom A attractors of  $C^2$  endomorphisms. Comparing with the case of diffeomorphisms, the major difficulty arises from non-invertibility. To overcome this deficiency, the inverse limit space has to be introduced. Notice that, when the inverse limit space is introduced, one can compare a full orbit with a sample orbit from random iteration of diffeomorphisms. Keep this in mind, with some necessary modifications, many ideas and techniques developed for the random diffeomorphisms, for which a systematic treatment is now available in [51], can be applied to our present study. The result was given by Qian and Zhu in [73], and we provide a detailed presentation in this chapter.

In Chapter VIII, we study the ergodic hyperbolic attractors. This chapter follows from the work of Jiang and Qian [28].

Chapters IX and X may be viewed as the climax of this book. In Chapter IX, we present here a generalized entropy formula for any Borel probability measure invariant under a  $C^2$  endomorphism. It is a non-invertible endomorphisms version of a formula obtained by Ledrappier and Young [43], hence covers theirs as a consequence. The generalized entropy formula relates closely to Eckmann-Ruelle conjecture for the endomorphism version; In Chapter X, we apply this entropy formula to hyperbolic measures preserved respectively by expanding maps and diffeomorphisms, proving Eckmann-Ruelle conjecture in these two situations. These two chapters are rewritten from the work of Qian and Xie [71] and Liu and Xie [54]. The proof of Eckmann-Ruelle conjecture (for diffeomorphisms) was first presented

by Barreira et al [7] (see also [64, pp. 279–292]); our proof is slightly different from theirs and seems more accessible.

In Appendix A, we show that Pesin entropy formula still holds true for  $C^2$  random endomorphisms if the sample measures of the invariant measure are smooth. This result covers those obtained by Pesin [63] for  $C^2$  diffeomorphisms, Liu [46] for  $C^2$  endomorphisms, Ledrappier and Young [44] for i.i.d. random diffeomorphisms and Liu [48] for two-sided stationary random endomorphisms.

In Appendix B, we present a large deviation theorem, it was included as one chapter in the first draft of the manuscript. Since the presentation is not self-contained, we prefer to shift it to the end as Appendix B. This part of the manuscript was prepared by Y. Zhao, see [53].

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