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Boundary Value Problems and Markov Processes

Second Edition



Springer

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To the memory of
Professor Kiyosi Itô
(1915–2008)

Preface to the Second Edition

This monograph is an expanded and revised version of a set of lecture notes for the graduate courses given by the author both at Hiroshima University (1995–1997) and at the University of Tsukuba (1998–2000) which were addressed to the advanced undergraduates and beginning-graduate students with interest in functional analysis, partial differential equations and probability.

The first edition of this monograph, which was based on the lecture notes given at the University of Tsukuba (1988–1990), was published in 1991. This edition was found useful by a number of people, but it went out of print after a few years.

This second edition has been revised to streamline some of the analysis and to give better coverage of important examples and applications. The errors in the first printing are corrected thanks to kind remarks of many friends. In order to make the monograph more up-to-date, additional references have been included in the bibliography.

This second edition may be considered as a short introduction to the more advanced book “*Semigroups, boundary value problems and Markov processes*” which was published in the Springer Monographs in Mathematics series in 2004. For graduate students working in functional analysis, partial differential equations and probability, it may serve as an effective introduction to these three interrelated fields of analysis. For graduate students about to major in the subject and mathematicians in the field looking for a coherent overview, it will provide a method for the analysis of elliptic boundary value problems in the framework of L^p spaces.

My special thanks go to the editorial staffs of Springer-Verlag for their unfailing helpfulness and cooperation during the production of this second edition.

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Last but not least, I owe a great debt of gratitude to my family who gave me moral support during the preparation of this book.

Tsukuba,
March 2009

Kazuaki Taira

Preface to the First Edition

This monograph is devoted to the functional analytic approach to a class of *degenerate* boundary value problems for second-order elliptic differential operators which includes as particular cases the Dirichlet and Neumann problems. We prove that this class of boundary value problems provides a new example of *analytic semigroups* both in the L^p topology and in the topology of uniform convergence. As an application, we show that there exists a strong *Markov process* corresponding to such a diffusion phenomenon that either absorption or reflection phenomenon occurs at each point of the boundary. Furthermore, we study a class of initial-boundary value problems for *semilinear* parabolic differential equations.

This monograph is an expanded version of a set of lecture notes for the graduate courses given by the author at the University of Tsukuba between 1988 and 1990. We confined ourselves to the simple boundary condition. This makes it possible to develop our basic machinery with a minimum of bother and the principal ideas can be presented concretely and explicitly. I hope that this monograph will lead to a better insight into the study of three interrelated subjects: elliptic boundary value problems, analytic semigroups and Markov processes. For additional information on many of the topics discussed here, I would like to call attention to my previous book *Diffusion Processes and Partial Differential Equations*, Academic Press, 1988.

I would like to express my hearty thanks to many colleagues and graduate students in my courses, whose helpful criticisms of my lectures resulted in a number of improvements. The manuscript is typeset in a camera-ready form using $\mathcal{A}\mathcal{M}\mathcal{S}$ - \TeX .

Tsukuba,
April 1991

Kazuaki Taira

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