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The Dirac Spectrum

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Dedicated to my dear and loving mother

Preface

This overview is based on the talk [101] given at the mini-workshop 0648c "*Dirac operators in differential and non-commutative geometry*", Mathematisches Forschungsinstitut Oberwolfach. Intended for non-specialists, it explores the spectrum of the fundamental Dirac operator on Riemannian spin manifolds, including recent research and open problems. No background in spin geometry is required; nevertheless the reader is assumed to be familiar with basic notions of differential geometry (manifolds, Lie groups, vector and principal bundles, coverings, connections, and differential forms). The surveys [41, 132], which themselves provide a very good insight into closed manifolds, served as the starting point. We hope the content of this book reflects the wide range of findings on and sometimes amazing applications of the spin side of spectral theory and will attract a new audience to the topic.

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Contents

Introduction

"Find a first order linear differential operator on $\mathbb{R}^n$ whose square coincides with the Laplace operator $-\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$." Give this as exercise to a group of undergraduates. If they can solve it for $n = 1$ then they have heard of complex numbers. If they can do it for $n \geq 2$ then either they believe to have solved it, or they claim to be students, or they know about Dirac.

For this simple-minded question and its rather involved answer lie at the origin of the whole theory of Dirac operators. It was P. Dirac who introduced [81] the operator now bearing his name when looking for an equation describing the probability amplitude of spin-$\frac{1}{2}$-particles (fermions, e.g. electrons) and that would fit into the framework of both special relativity and quantum mechanics. Mathematically formulated, his problem consisted in finding a square root of the Klein-Gordon operator (d'Alembert plus potential) on the 4-dimensional Minkowski spacetime. It already came as a breakthrough when Dirac showed that the problem could be solved not for the scalar operator but for the $\mathbb{C}^2$-valued one using the so-called Pauli matrices as coefficients.

Like many objects invented by physicists, the Dirac operator was soon called upon to develop an own mathematical life. It was indeed later discovered that the setup of Clifford algebras allowed it to be defined in a general geometrical framework on "almost any" smooth semi-Riemannian manifold. Here "almost any" means that there exists a topological restriction on the manifold for the Dirac operator to be well-defined - the spin condition, see Chapter 1 - which is however satisfied on most "known" manifolds. This mathematical investigation gave birth to spin geometry. One of the first and probably most famous achievements of spin geometry was the discovery of a topological obstruction to positive scalar curvature as a relatively straightforward application of the Atiyah-Singer-Index theorem, see Chapter 3 and references therein.

It would be very modest to claim that spin geometry has remained lively since. Less than twenty years after Atiyah and Singer's breakthrough, the whole mathematical community could only gape at E. Witten's amazingly simple proof of the positive mass theorem based on the analysis of a Dirac-type operator of a bounding hypersurface [235]. At the same time, noncommutative geometry made the Dirac operator one of its keystones as

it allows to reconstruct a given Riemannian spin manifold from its so-called canonical spectral triple [78, 112]. Independently, special eigenvectors of the Dirac operator called Killing spinors have become some of the physicists' main tools in the investigation of supersymmetric models for string theory in dimension 10, see e.g. [184]. In a more geometrical context, Dirac-type operators have been successfully applied in as varied situations as finding obstructions to minimal Lagrangian embeddings [141], rigidity issues in extrinsic geometry [138] or the Willmore conjecture [14, 17], just to cite a few of them.

Exploring the spin geometrical aspects of all the above-mentioned topics would require a small encyclopedia, therefore we focus on a particular one. Out of lack of up-to-date literature on the subject, we choose to deal in this book with the spectrum of the Dirac operator on complete (mainly compact) Riemannian spin manifolds with or without boundary. In particular we do not intend to give any kind of extensive introduction to spin geometry, see [63, 88] and the mother-reference [173] in this respect (the physics-oriented reader may prefer [227]). Since it was not possible to handle all facets of the Dirac spectrum in one volume, we had to leave some of them aside. To keep the book as self-contained as possible, we sketch those briefly in the last chapter.

We begin with introducing the Dirac operator and its geometrical background. Although the definition is rather involved, we try to remain as simple as possible so as not to drown the reader in technical considerations such as representation theory of Clifford algebras or the topological spin condition. In Chapter 1 we define the spin group, spin structures on manifolds, spinors (which are sections of a vector bundle canonically attached to manifolds carrying a spin structure) and the Dirac operator acting on spinors. We show that the Dirac operator is an elliptic, formally self-adjoint linear differential operator of first order and, if the underlying Riemannian manifold is furthermore complete, then it is essentially self-adjoint in L^2. In particular, if the manifold is closed, then the spectrum of its Dirac operator is well-defined, real, discrete and unbounded. In case the boundary of the manifold is non-empty, elliptic boundary conditions have to be precised for the spectrum to be well-defined and discrete.
At this point we underline that only a so-called spin^c structure is needed on the manifold in order for the Dirac operator to be well-defined. Spin^c structures require weaker topological assumptions to exist than spin structures. Since however their treatment would bring us too far, we choose to ignore them in this book (see Section 8.4 for references).

The second chapter deals with examples of closed manifolds whose Dirac spectrum - or at least some eigenvalues - can be explicitly computed. They all belong to the class of homogeneous spaces, for which we recall the representation-theoretical method allowing one to describe the Dirac operator as a family of matrices, see Theorem 2.2.1.

Since it would be illusory to aim at the explicit knowledge of the Dirac spectrum in general, one way for studying it consists in estimating the eigenvalues. In Chapter 3 we consider an arbitrary closed Riemannian spin manifold and describe the main lower bounds that have been proved for its Dirac spectrum. Almost all of them rely on the Schrödinger-Lichnerowicz formula (1.15) and thus involve the scalar curvature of the manifold. Starting from the most general estimate - Friedrich's inequality (3.1) - we show how it can be improved in some particular cases. The equality-case of most of those inequalities is characterized by the existence of special sections (e.g. Killing spinors) which give rise to interesting geometrical features. We shift the treatment of some of them to Appendix A since they are of independent interest.

In the situation where the manifold has a non-empty boundary, we consider four different boundary conditions, two of which generalize those originally introduced by Atiyah, Patodi and Singer [27]. We describe in Chapter 4 the corresponding lower bounds *à la Friedrich* that have been obtained in this context.

The techniques involved for proving lower bounds drastically differ from those used in getting upper eigenvalue bounds. In the latter case - and if the manifold is closed - there exist two methods available for the Dirac operator, the first one based on index theory and the second one on the min-max principle. Chapter 5 collects the different geometrical upper bounds that have been proved with the help of those, separating the intrinsic - depending on the intrinsic geometry only - from the extrinsic ones, i.e., depending on some map from the manifold into another one.

In Chapter 6, we turn to the closely related issues of isospectrality and prescription of eigenvalues. In a first part, we discuss isospectrality results obtained on spaceforms of non-negative curvature and on circle bundles. Turning to the eigenvalue 0, we detail in Section 6.2 existence as well as non-existence results for harmonic spinors, i.e., sections lying in the kernel of the Dirac operator. Here there is a remarkable difference between dimensions 2 and greater than 2. We end this chapter with a brief account on how the lower part of the spectrum can always be prescribed provided it does not contain 0.

On non-compact Riemannian spin manifolds another part of the spectrum beside the eigenvalues must be taken into account, the so-called continuous spectrum. For the Dirac operator it is well-defined as soon as the underlying Riemannian manifold is complete, however the square of the Dirac operator always has a spectrum (see Section 7.1). Only few examples are known where the whole Dirac spectrum can be computed. In Chapter 7 we mainly discuss the interactions between the geometry or topology of the manifold with the Dirac spectrum, in particular we focus on whether it can be purely discrete or continuous.