

# Lecture Notes in Mathematics

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# Lectures on Topological Fluid Mechanics

Lectures given at the  
C.I.M.E. Summer School  
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# Preface

It seems very appropriate to publish these *Lectures* in the 150th anniversary of the publication of Helmholtz's seminal paper on vortex motion (1858), that may be regarded as the pioneering work on fundamental questions in topological fluid mechanics. The field is going through a period of great revival, benefitting from the formidable progress in knot theory and differential topology, on the one hand, and mathematical and computational fluid dynamics, on the other. It is therefore with great pleasure that I warmly thank the contributors to this volume for providing such interesting collection of valuable research papers. All the material presented here is actually an update on the original material presented in the lecture notes delivered by six of us, on the occasion of a CIME Summer School (of the Unione Matematica Italiana) in 2001, a School that I had the honour and pleasure to organize and direct in Cetraro, a charming location on the rugged coastline of southern Italy.

The Summer School was qualitatively very successful, thanks also to the many attendees, who themselves were renown experts in their own fields, coming from UK, Germany, Russia, USA, Japan, Tunisia, Ukraine, Canada and, of course, Italy! Given this luxury, as Editor-in-charge of the proceedings, I couldn't resist to break one of CIME's most consolidated rules, by inviting one of the attendees, *Patrick Bangert*, as a representative of such a qualified audience, to collect some background information for an introductory chapter to knot and braid theory. The material of **Braids and knots** (71-page long!) is therefore an invaluable, complementary addition to the lecture programme of 2001, covering elements of braid theory and knot polynomials, as well as more advanced aspects of braid and knot classification, including up-dates on the Word Problem, the Conjugacy Problem and Markov's Theorem. **Topological quantities: calculating winding, writhing, linking and higher order invariants**, by *Mitch Berger*, is on winding number techniques to calculate topological numbers for both closed and open curves, with applications to magnetic fields in magnetohydrodynamics. **Tangles, rational knots and DNA**, by *Lou Kauffman* and *Sofia Lambropoulou*, is dedicated to the theory of tangles and unoriented and oriented rational knots, with applications

to DNA recombination processes. **The group and Hamiltonian descriptions of hydrodynamical systems** is due to *Boris Khesin*, who reviews applications of Hamiltonian approach and group theory to ideal fluid dynamics and integrable systems, with particular emphasis on the Landau-Lifschitz and Korteweg-de Vries equations. *Keith Moffatt* presents and discusses in **Singularities in fluid dynamics and their resolution** three types of singularity that can arise in fluid dynamical problems: (i) singularities driven by boundary motion in conjunction with viscosity; (ii) free-surface (cusp) singularities associated with surface-tension and viscosity; (iii) interior point singularities of vorticity associated with intense vortex stretching. In paying tribute to 150 years of topological fluid mechanics, *Renzo Ricca* (**Structural complexity and dynamical systems**) reviews Helmholtz's (1858) original contributions on topological aspects in fluid dynamics, presenting, then, a overview on current work on structural complexity analysis and some new results on topological bounds on magnetic energy of knots and links and helicity-crossing number relations. Finally, in **Random knotting: theorems, simulations and applications**, *De Witt Summers* presents a very nice survey on random knotting and topological entanglement of filaments in space, starting from work on the Frisch-Wasserman-Delbruck Conjecture, and concluding with applications to viral DNA molecule packing and knotting. The volume is therefore a rather wide-ranging collection of important themes in current topological fluid mechanics and I can only hope that the interested reader may find here further stimulus to his/her own research work.

For all of this, I am particularly indebted, first of all to the authors, and secondly to the former Director of CIME, Arrigo Cellina, and to the present Director, Pietro Zecca, who patiently trusted in the aim of this project. I also want to thank Carla Dionisi for her technical support and last, but not least, Springer-Verlag, at Heidelberg.

University of Milano-Bicocca,  
March, 2008

*Renzo L. Ricca*

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