Lecture Notes in Mathematics 1969

Bernard Roynette · Marc Yor

Penalising Brownian Paths

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Bernard Roynette
Université Nancy I
Institut Elie Cartan
Faculté des Sciences
Département de Mathématiques
B.P. 239
54506 Vandoeuvre-les-Nancy CX
France
roynette@iecn.u-nancy.fr

Marc Yor
Université Paris VI
Laboratoire de Probabilités
175 rue de Chevaleret
75013 Paris
France

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Preface

Since roughly 2002, we have been interested in establishing a number of *penalisation results* for Brownian motion.

• Let us first explain the term "penalisation":
by putting an adequate weight Γ_t on the Wiener measure W, constructed on $(\Omega = C(\mathbb{R}_+, \mathbb{R}), (X_t, \mathcal{F}_t)_{t\geq 0})$, where $X_s(\omega) = \omega(s)$, $\omega \in \Omega$, $s \geq 0$, denotes the canonical process, and $(\mathcal{F}_t = \sigma\{X_s, s \leq t\}, t \geq 0)$ its natural filtration, we wish to show that :

$$W_t^\Gamma := \Gamma_t \bullet W \tag{P1}$$

when restricted to $\mathcal{F}_s$, for any finite s, converges as $t \to \infty$, i.e. :

$$\forall s > 0\,,\ \forall F_s \in \mathcal{F}_s\,,\ W_t^\Gamma(F_s) \underset{t\to\infty}{\longrightarrow} W_\infty^\Gamma(F_s) \tag{P2}$$

Assuming that this holds, it is not difficult to show that W_∞^Γ induces a probability on $(\Omega, \mathcal{F}_\infty)$, such that :

$$\forall s > 0\,, W^\Gamma_{\infty|\mathcal{F}_s} = M_s \bullet W_{|\mathcal{F}_s} \tag{P3}$$

for a certain martingale (M_s) with respect to $(W, (\mathcal{F}_s))$.
We then say that we have penalised W with the weight process $(\Gamma_t, t \geq 0)$; for example, taking $\Gamma_t = \varphi(\sup\limits_{u\leq t} X_u)/W(\varphi(\sup\limits_{u\leq t} X_u))$, for bounded φ's with compact support, we obtain the existence of W_∞^Γ under which $\sup\limits_{u\geq 0} X_u$ is finite.
Thus, the process (X_t), under W_∞^Γ has a radically different behavior than under W, namely the supremum process has been penalised so that it becomes finite valued under the new probability measure W_∞^Γ.
We have been looking systematically for such alterations of the Wiener measure, by taking weight processes involving the supremum, or the local time at 0, or..., or in dimension 2, the winding process of planar Brownian motion...

Besides these examples, the most natural penalisations of W, consist to take :

$$\Gamma_t = \exp\left(-\int_0^t q(X_s)ds\right) / E_W\left(\exp\left(-\int_0^t q(X_s)ds\right)\right) \tag{P4}$$

for some integrable function $q : \mathbb{R} \to \mathbb{R}_+$.
We call these Feynman-Kac penalisations.
• We now explain about the organisation of these Lecture notes :

- **Chapter 0**, is a detailed Introduction to the whole volume, including some discussion comparing penalisation and other topics, e.g. : the small ball problem...

- **Chapter 1**, a version of which has been published in Japanese Journal of Mathematics (2006) develops a number of cases of penalisations, together with the presentation of the relevant prerequisites for Brownian motion, e.g. : the definition of Brownian local times, and so on...
Each of the cases presented there has been the subject of a fully developed paper, a series of which have appeared in Studia Math. Hung. (see [RVY, i], $i = \mathrm{I}, \mathrm{II}, \dots \mathrm{VII}$ and [RY, j], $j = \mathrm{VIII}, \mathrm{IX}$, in the Bibliography, p. 34).

- **In Chapter 2**, we take up the study of Feynman-Kac penalisations, but there, we look for a global approach, as follows :
we show the existence of a σ-finite measure $\boldsymbol{\Lambda}$ on $C(\mathbb{R}, \mathbb{R}_+)$ such that, for conveniently integrable functionals $\Gamma(L_t^y; y \in \mathbb{R})$, we have :

$$\sqrt{t}\, W(\Gamma(L_t^y; y \in \mathbb{R})) \underset{t\to\infty}{\longrightarrow} \boldsymbol{\Lambda}(\Gamma)\ .$$

Our aim then is to show that, associated with these functionals, there is a penalisation result, with limiting martingales (M_s) as in formula (P3), expressible in terms of the measure $\boldsymbol{\Lambda}$.

- **In Chapter 3**, we consider another general framework of penalisations of W with, for example, functions of the sequence :

$$(V_t^{(1)}, V_t^{(2)}, ..., V_t^{(n)}, ...)$$

of lengths of Brownian excursions, away from 0, up to time t, ranked in decreasing order, as in the studies of Pitman-Yor related to the Poisson-Dirichlet distributions (see [PY_5], references at the end Chapter 3).
Again, a preliminary study, involving only $V_t^{(1)}$ has been made in [RVY,VII] (see the end of Chapter 1), whereas here we show the existence of a σ-finite measure $\boldsymbol{\Pi}$ much as in Chapter 2, from which we shall be able to describe the general penalisations obtained from these ranked lengths of excursions.

- In Chapter 4, we question in broad terms the validity of our discussion in the following sense :
in our asymptotic studies, we always restricted ourselves to fixed finite intervals $[0, s]$, i.e. : we looked for the existence of the limit of

$$W_t^\Gamma(F_s)\,,\ \text{as } t \to \infty\,,\ \text{for } F_s \in \mathcal{F}_s\ .$$

In this Chapter 4, we ask about the closeness of $W^\Gamma_{\infty|\mathcal{F}_t}$ and of $\Gamma_t \bullet W_{|\mathcal{F}_t}$, as $t \to \infty$, and we are able to show that they are far apart, in that the total variation of the difference converges to a positive constant.
A number of related questions are also examined.

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As a temporary conclusion of our penalisation studies, let us look at what has or has not been achieved :
• we have shown the existence of a penalised limiting measure for W_t^Γ, as $t \to \infty$, for a large class of interesting and "natural" weight processes. On the other hand, we have not been able to find general criterion on such processes that would ensure the existence of this limit;
• even if one is not a priori interested in penalisations per se, these studies are a source of "creation" of Brownian martingales, which may be of interest by themselves; see in particular the discussion in Chapter 0.
• more complicated weight processes have been considered in the probabilistic - statistical mechanics oriented - literature, involving instead of the simple integrals

$$\int_0^t q(X_s)ds$$

$$\text{e.g :} \qquad \text{double integrals : } \int\!\!\int_{[0,t]^2} q(X_u - X_s)du\,ds \tag{P5}$$

In fact, Symanzyk's program - which is closely related to Edwards' model - consists in looking for the existence of limiting results as $n \to \infty$, for fixed t, for the normalized weights :

$$\exp\left(-\int\!\!\int_{[0,t]^2} q_n(X_u - X_s)du\,ds\right)$$

where : $q_n(x) = n^d\, q(nx) \quad (x \in \mathbb{R}^d)$, with q integrable on $\mathbb{R}^d$, and W the law of d-dimensional Brownian motion.

We hope that, in some near future, some of the techniques we have developed throughout this monograph may be of some use for these more complicated penalisations.

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A few notable features of this volume

i) Certain σ-finite measures, with infinite total mass, play an important role in the description of our results. These measures are always denoted in **bold** character, in order to *"separate"* them from our more common probability measures, indicated in plain character; examples : $\mathbf{W}_x$, $\mathbf{\Lambda}_x$, in contrast to $\Lambda_\ell^\pm$, $P_x^{(t)}$,...

ii) We have made sure that each of the five chapters may be read independently from the others; this is quite natural, as each topic can be developed from the use of adequate tools. Thus, the reader may browse through the volume easily, and be attracted first by Chapter 4 say, then come back to Chapter 3 and so on, ..., without difficulty. *Consequently, each Chapter ends with its own bibliography.* However, throughout the volume, all references have been homogenized, i.e : reference [R] in Chapter 1 is also reference [R] in Chapter 3, if it plays some role there... We thank Jim Pitman for asking us to complete our references, with respect both to "classical works" and to our more recent understanding of penalisations, e.g. : in Chapter 2, we have sketched some results obtained between April and June 2007 jointly with J. Najnudel.

iii) A priori, penalisation studies might be developed for a fairly large class of stochastic processes; however, in this volume, our processes of reference are either Brownian motion, or Bessel processes of dimension $d \in (0, 2)$. The reader shall find the main properties which are shared/enjoyed by these processes and are of interest in our penalisation studies discussed in Chapter 1 in the form of specific $\boxed{\text{Items}}$.

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- Pierre Vallois, who undertook a large part of these penalisation studies with us, as our "Roman Number" series clearly shows;
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This Monograph is dedicated to
Frank Knight (1933-2007).

His study of Taboo Processes in 1969 is a beautiful example of penalisation.