Part I
Categorical and Operadic Background

Foreword: Categorical Conventions

0.1 Functor Categories. In this book, we deal with categories of functors $F : \mathcal{A} \to \mathcal{X}$. Generally, the category $\mathcal{A}$ is not supposed to be small, but to avoid set-theoretic difficulties we assume tacitely that any category $\mathcal{A}$ considered in the book contains a small subcategory $\mathcal{A}_f$ such that every object $X \in \mathcal{A}$ is the filtered colimit of a diagram of $\mathcal{A}_f$*. Moreover, we consider tacitely only functors $F : \mathcal{A} \to \mathcal{X}$ that preserve filtered colimits and the notation $\mathcal{F}(\mathcal{A}, \mathcal{X})$ refers to the category formed by these functors. All functors which arise from our constructions satisfy this assumption. Under this convention, we obtain that the category $\mathcal{F}(\mathcal{A}, \mathcal{X})$ has small morphism sets and no actual set-theoretic difficulty occurs.

Besides, the existence of the small category $\mathcal{A}_f$ implies that a functor $\phi_! : \mathcal{A} \to \mathcal{X}$ admits a right adjoint $\phi^* : \mathcal{X} \to \mathcal{A}$ if and only if it preserves colimits, because the set-theoretic condition of the adjoint functor theorem is automatically fulfilled.

0.2 Notation for Colimits. Categories occur at two levels in our constructions: we use ground categories, which are usually symmetric monoidal categories, and categories of algebras over operads, which lie over an underlying ground category. To distinguish the role of these categories, we use two system of conventions to represent colimits: we adopt additive notation (0 for the initial object and $\oplus$ for the coproduct) for ground categories, and the base-set notation $\vee$ for the coproduct in categories of algebras over operads. Nevertheless, we return to base-set (or set) notation in particular instances of ground categories (sets, simplicial sets, topological spaces, ...) for which this convention is usual.

The base-set notation is applied to the category of operads in a base symmetric monoidal category (see §3.1.1). The base-set notation is also used for categories to which no role is assigned.

Note that ground categories are not assumed to be additive in general. The initial object 0 is not supposed to be a zero object, and the coproduct $\oplus$ is not supposed to be a bi-product.

* We only make an exception for the category of topological spaces.

0.3 Symmetric Monoidal Categories and Enriched Categories. The structure of a symmetric monoidal category gives the categorical background of the theory of operads. The definition of this notion is reviewed in the next chapter. For the moment, recall briefly that a symmetric monoidal category consists of a category $\mathcal{C}$ equipped with a tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and a unit object $\mathtt{1} \in \mathcal{C}$ which satisfies the usual unit, associativity and symmetry relations of the tensor product of modules over a ring.

The notation $\mathrm{Mor}_{\mathcal{E}}(X, Y)$ is used throughout the book to refer to the morphism-sets of any category $\mathcal{E}$. But many categories are assumed to be enriched over a base symmetric monoidal category and come equipped with a hom-bifunctor with value in $\mathcal{C}$, if $\mathcal{C}$ refers to the base category (see §1.1.12). The hom-objects of every enriched category $\mathcal{E}$ are denoted by $\mathit{Hom}_{\mathcal{E}}(X, Y)$, to be distinguished from the morphism sets $\mathrm{Mor}_{\mathcal{E}}(X, Y)$.

0.4 Point-Set Symmetric Monoidal Categories. We illustrate our constructions by applications in categories of modules over a commutative ground ring, in categories of differential graded modules (dg-modules for short), in categories of Σ_*-objects (also called symmetric objects in English words), or in categories of right modules over an operad. In these instances of symmetric monoidal categories, the tensor product $X \otimes Y$ is spanned in a natural sense by certain tensors $x \otimes y$, where $(x, y) \in X \times Y$, and any morphism $\phi : X \otimes Y \to Z$ is equivalent to a kind of multilinear map $\phi : x \otimes y \mapsto \phi(x, y)$ on the set of generating tensors. The tensor product $\theta : (x, y) \mapsto x \otimes y$ represents itself a universal multilinear map $\theta : X \times Y \to X \otimes Y$. The representation of morphisms $\phi : X \otimes Y \to Z$ by actual multilinear maps can be extended to homomorphisms, elements of internal hom-objects of these categories. This pointwise representation, usual for modules over a ring, is formalized in §1.1.5 in the context of dg-modules, in §2.1.9 in the context of Σ_*-objects, and in §6.1.3 in the context of right modules over an operad.

In illustrations, we apply the pointwise representation of tensors to the categories, derived from the category of modules, which are mentioned in this paragraph. But, of course, pointwise representations of tensors hold in the category of sets, whose tensor product is defined by the cartesian product, and more generally in any point-set category derived from the cartesian category of sets, like the category of topological spaces or the category simplicial sets.

In the sequel, we speak abusively of a point-set context to refer to a symmetric monoidal category in which a pointwise representation of tensors holds.

0.5 The Principle of Generalized Point-Tensors. The pointwise representation, which makes many definitions more basic, is used in applications. To simplify, we may make explicit the example of modules over a ring only and we may omit the case of dg-modules (respectively, Σ_*-objects, right modules over operads) in illustrations. Usually, the generalization from modules to dg-modules (respectively, Σ_*-objects, right modules over operads) can be carried out automatically within the pointwise representation, without calling back the category formalism.

The only rule is to keep track of tensor permutations in mappings to replace a tensor $x \otimes y$ by a decorated tensor $\sigma \cdot x \otimes y$, where the decoration σ consists of a sign in the context of dg-modules (see §1.1.5), a block permutation in the context of Σ_*-objects and modules over operads (see §2.1.9), a sign combined with a block permutation in the context of symmetric objects in dg-modules ... The rule arises from the definition of the symmetry isomorphism $\tau : X \otimes Y \xrightarrow{\simeq} X \otimes Y$ in these symmetric monoidal categories.

To refer to these rules, we say that we apply the principle of generalized point-tensors.