

Lecture Notes in Mathematics

1962

Editors:

J.-M. Morel, Cachan

F. Takens, Groningen

B. Teissier, Paris

Robert Dalang · Davar Khoshnevisan · Carl Mueller
David Nualart · Yimin Xiao

A Minicourse on Stochastic Partial Differential Equations

Editors:

Davar Khoshnevisan
Firas Rassoul-Agha

 Springer

Authors and Editors

Robert C. Dalang
Institut de Mathématiques
Ecole Polytechnique Fédérale de Lausanne
Station 8
1015 Lausanne
Switzerland
robert.dalang@epfl.ch

Carl Mueller
Department of Mathematics
University of Rochester
Rochester, NY 14627
USA
cmlr@math.rochester.edu

Firas Rassoul-Agha
Department of Mathematics
University of Utah
Salt Lake City, UT 84112-0090
USA
firas@math.utah.edu

Davar Khoshnevisan
Department of Mathematics
University of Utah
Salt Lake City, UT 84112-0090
USA
davar@math.utah.edu

David Nualart
Department of Mathematics
University of Kansas
Lawrence, KS 66045
USA
nualart@math.ku.edu

Yimin Xiao
Department of Statistics and Probability
Michigan State University
A-413 Wells Hall
East Lansing, MI 48824
USA
xiao@stt.msu.edu

ISBN 978-3-540-85993-2 e-ISBN 978-3-540-85994-9
DOI: 10.1007/978-3-540-85994-9

Lecture Notes in Mathematics ISSN print edition: 0075-8434
ISSN electronic edition: 1617-9692

Library of Congress Control Number: 2008934459

Mathematics Subject Classification (2000): 60H15, 60H20, 60H25

© 2009 Springer-Verlag Berlin Heidelberg

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilm or in any other way, and storage in data banks. Duplication of this publication or parts thereof is permitted only under the provisions of the German Copyright Law of September 9, 1965, in its current version, and permission for use must always be obtained from Springer. Violations are liable to prosecution under the German Copyright Law.

The use of general descriptive names, registered names, trademarks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

Cover design: SPi Publishing Services

Printed on acid-free paper

9 8 7 6 5 4 3 2 1

springer.com

Preface

From May 8 to May 19th of 2006, the Department of Mathematics at the University of Utah hosted a minicourse on some modern topics in stochastic partial differential equations [SPDEs]. The participants included graduate students and recent PhDs from across North America, as well as research mathematicians at diverse stages of their careers. Intensive courses were given by Robert C. Dalang, Davar Khoshnevisan, An Le, Carl Mueller, David Nualart, Boris Rozovsky, and Yimin Xiao. The present book is comprised of most of those lectures.

For nearly three decades, the topic of SPDEs has been an area of active research in pure and applied mathematics, fluid mechanics, geophysics, and theoretical physics. The theory of SPDEs has a similar flavor as PDEs and interacting particle systems in the sense that most of the interesting developments generally evolve in two directions: There is the general theory; and then there are specific problem-areas that arise from concrete questions in applied science. As such, it is unlikely that there ever will be a cohesive all-encompassing theory of stochastic partial differential equations. With that in mind, the present volume follows the style of the Utah minicourse in SPDEs and attempts to present a selection of interesting themes within this interesting area. The presentation, as well as the choice of the topics, were motivated primarily by our desire to bring together a combination of methods and deep ideas from SPDEs (Chapters 1, 2, and 4) and Gaussian analysis (Chapters 3 and 5), as well as potential theory and geometric measure theory (Chapter 5). Ours is a quite novel viewpoint, and we believe that the interface of the mentioned theories is fertile ground that shows excellent potential for continued future research.

We are aware of at least four books on SPDEs that have appeared since we began to collect the material for this project [4; 8; 12; 14]. Although there is little overlap between those books and the present volume, the rapidly-growing number of books on different aspects of SPDEs represents continued, as well as a growing, interest in both the theory as well as the applications of the subject. The reader is encouraged to consult the references for examples

in: (i) Random media [2; 4; 18] and filtering theory [15]; (ii) applications in fluid dynamics and turbulence [1; 2; 17]; and (iii) in statistical physics of disordered media [2; 6; 7; 10]. Further references are scattered throughout the lectures that follow. The reader is invited to consult the references to this preface, together with their voluminous bibliographies, for some of the other viewpoints on this exciting topic.

The Utah Minicourse on SPDEs was funded by a generous VIGRE grant by the National Science Foundation, to whom we are grateful. We thank also the lecturers and participants of the minicourse for their efforts. Finally, we extend our wholehearted thanks to the anonymous referee; their careful reading and thoughtful remarks have led to a more effective book.

Salt Lake City, Utah
July 1, 2008

Davar Khoshnevisan
Firas Rassoul-Agha

References

- [1] A. Bourlioux, A. J. Majda, and O. Volkov (2006). Conditional statistics for a passive scalar with a mean gradient and intermittency. *Phys. Fluids* **18**(10), 104102, 10 pp.
- [2] René A. Carmona and S. A. Molchanov (1994). Parabolic Anderson problem and intermittency, *Mem. Amer. Math. Soc.* **108**(518)
- [3] *Stochastic Partial Differential Equations: Six Perspectives* (1999). Edited by Rene A. Carmona and Boris Rozovskii, American Math. Society, Providence, Rhode Island
- [4] Pao-Liu Chow (2007). *Stochastic Partial Differential Equations*. Chapman & Hall/CRC, Boca Raton, Florida
- [5] Gopinath Kallianpur and Jie Xiong (1995). *Stochastic Differential Equations in Infinite Dimensional Spaces*. Institute of Math. Statist. Lecture Notes—Monograph Series, Hayward, California
- [6] Mehran Kardar (1987). Replica Bethe ansatz studies of two-dimensional interfaces with quenched random impurities. *Nuclear Phys.* B290 [FS20], 582–602
- [7] Mehran Kardar, Giorgio Parisi, and Yi-Cheng Zhang (1986). Dynamic scaling of growing interfaces, *Phys. Rev. Lett.* **56**(9), 889–892
- [8] Peter Kotelenetz (2008). *Stochastic Ordinary and Stochastic Partial Differential Equations*, Springer, New York
- [9] Nicolai V. Krylov (2006). On the foundations of the L_p -theory of stochastic partial differential equations. In: *Stochastic Partial Differential Equations and Applications—VII*, 179–191, Lecture Notes Pure Appl. Math., Chapman & Hall/CRC, Boca Raton, Florida
- [10] Ernesto Medina, Terrence Hwa, and Mehran Kardar (1989). Burgers equation with correlated noise: Renormalization-group analysis and applications to directed polymers and interface growth. *Phys. Rev. A*, **38**(6), 3053–3075

- [11] Pierre-Louis Lions and Panagiotis Souganidis (2000). Fully nonlinear stochastic pde with semilinear stochastic dependence. *C.R. Acad. Sci. Paris Sér. I Math.* **331**(8), 617–624
- [12] S. Peszat and J. Zabczyk (2007). *Stochastic Partial Differential Equations with Lévy Noise*. Encyclopedia of Mathematics and Its Applications, Cambridge University Press, Cambridge
- [13] Guiseppe Da Prato and Jerzy Zabczyk (1992). *Stochastic Equations in Infinite Dimensions*, Encyclopedia of Mathematics and Its Applications, Cambridge University Press, Cambridge
- [14] Claudia Prévôt and Michael Röckner (2007). *A Concise Course on Stochastic Partial Differential Equations*. Lecture Notes in Mathematics **1905**, Springer-Verlag, Berlin, Heidelberg
- [15] B. L. Rozovskii (1990). *Stochastic Evolution Systems*. Kluwer Academic Publishing Group, Dordrecht (Translated from the original Russian by A. Yarkho, *Math. and Its Applicatons* (Soviet Series), 35)
- [16] J. B. Walsh (1986). *An Introduction to Stochastic Partial Differential Equations*. In: Ecole d'Ete de Probabilites de Saint Flour XIV, Lecture Notes in Mathematics **1180**, 265–438
- [17] Wojbor A. Woyczyński (1998). *Burgers–KPZ Turbulence*, Lecture Notes in Mathematics **1700**, Springer, Berlin
- [18] Ya. B. Zeldovich, S. A. Molchanov, A. S. Ruzmaïkin, and D. D. Solokov (1987). Intermittency in random media, *Uspekhi Fiz. Nauk* **152**(1), 3–32 (in Russian) [English translation in *Soviet Phys. Uspekhi* **30**(5), 353–369, 1987]

Contents

A Primer on Stochastic Partial Differential Equations

<i>Davar Khoshnevisan</i>	1
1 What is an SPDE?	1
2 Gaussian Random Vectors	2
3 Gaussian Processes	2
4 Regularity of Random Processes	8
5 Martingale Measures	14
6 A Nonlinear Heat Equation	23
7 From Chaos to Order	32
References	36

The Stochastic Wave Equation

<i>Robert C. Dalang</i>	39
1 Introduction	39
2 The Stochastic Wave Equation	41
3 Spatially Homogeneous Gaussian Noise	47
4 The Wave Equation in Spatial Dimension 2	49
5 A Function-Valued Stochastic Integral	56
6 The Wave Equation in Spatial Dimension $d \geq 1$	58
7 Spatial Regularity of the Stochastic Integral ($d = 3$)	61
8 Hölder-Continuity in the 3-d Wave Equation	70
References	71

Application of Malliavin Calculus to Stochastic Partial Differential Equations

<i>David Nualart</i>	73
1 Introduction	73
2 Malliavin Calculus	73
3 Application of Malliavin Calculus to Regularity of Probability Laws	83
4 Stochastic Heat Equation	92

X Contents

5	Spatially Homogeneous SPDEs	99
	References	108

**Some Tools and Results for Parabolic Stochastic Partial
Differential Equations**

<i>Carl Mueller</i>	111
1 Introduction	111
2 Basic Framework	113
3 Duality	115
4 Large Deviations for SPDEs	125
5 A Comparison Theorem	129
6 Applications	131
References	142

**Sample Path Properties of Anisotropic Gaussian Random
Fields**

<i>Yimin Xiao</i>	145
1 Introduction	145
2 Examples and General Assumptions	148
3 Properties of Strong Local Nondeterminism	160
4 Modulus of Continuity	164
5 Small Ball Probabilities	168
6 Hausdorff and Packing Dimensions of the Range and Graph	170
7 Hausdorff Dimension of the Level Sets and Hitting Probabilities ..	183
8 Local Times and Their Joint Continuity	194
References	207

List of Participants	213
-----------------------------------	-----

Index	215
--------------------	-----

Contributors

Davar Khoshnevisan

Department of Mathematics
The University of Utah
Salt Lake City
UT 84112-0090, USA
davar@math.utah.edu
<http://www.math.utah.edu/~davar>

Robert C. Dalang

Institut de Mathématiques
Ecole Polytechnique Fédérale de
Lausanne
Station 8, CH-1015
Lausanne, Switzerland
robert.dalang@epfl.ch
<http://ima.epfl.ch/~rdalang/>

David Nualart

Department of Mathematics
University of Kansas
Lawrence, KS 66045, USA
nualart@math.ku.edu
<http://www.math.ku.edu/~nualart/>

Carl Mueller

Department of Mathematics
University of Rochester, Rochester
NY 14627, USA
cmlr@math.rochester.edu
<http://www.math.rochester.edu/~cmlr>

Yimin Xiao

Department of Statistics and
Probability
A-413 Wells Hall
Michigan State University
East Lansing
MI 48824, USA
xiao@stt.msu.edu
<http://www.stt.msu.edu/~xiaoyimi>