

# Lecture Notes in Mathematics

1961

## **Editors:**

J.-M. Morel, Cachan

F. Takens, Groningen

B. Teissier, Paris

Giuseppe Buttazzo · Aldo Pratelli  
Sergio Solimini · Eugene Stepanov

# Optimal Urban Networks via Mass Transportation

 Springer

Giuseppe Buttazzo  
Dipartimento di Matematica  
Università di Pisa  
Largo Bruno Pontecorvo 5  
56127 Pisa  
Italy  
buttazzo@dm.unipi.it

Aldo Pratelli  
Dipartimento di Matematica  
Università di Pavia  
Via Ferrata 1,  
27100 Pavia  
Italy  
aldo.pratelli@unipv.it

Sergio Solimini  
Dipartimento di Matematica  
Politecnico di Bari  
Via Amendola 126/b  
70126 Bari  
Italy  
solimini@dm.uniba.it

Eugene Stepanov  
St. Petersburg University  
of Information Technology, Mechanics  
and Optics  
Kronverkskij pr. 49  
197101 St. Petersburg  
Russia  
stepanov.eugene@gmail.com

ISBN: 978-3-540-85798-3 e-ISBN: 978-3-540-85799-0  
DOI: 10.1007/978-3-540-85799-0

Lecture Notes in Mathematics ISSN print edition: 0075-8434  
ISSN electronic edition: 1617-9692

Library of Congress Control Number: 2008935629

Mathematics Subject Classification (2000): 49J45, 49Q10, 49Q15, 49Q20, 90B06, 90B10, 90B20

© 2009 Springer-Verlag Berlin Heidelberg

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilm or in any other way, and storage in data banks. Duplication of this publication or parts thereof is permitted only under the provisions of the German Copyright Law of September 9, 1965, in its current version, and permission for use must always be obtained from Springer. Violations are liable to prosecution under the German Copyright Law.

The use of general descriptive names, registered names, trademarks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

*Cover design:* SPi Publisher Services

Printed on acid-free paper

9 8 7 6 5 4 3 2 1

springer.com

# Preface

The monograph is dedicated to a class of models of optimization of transportation networks (urban traffic networks or networks of railroads and highways) in the given geographic area. One assumes that the data on distributions of population and of services/workplaces (i.e. sources and sinks of the network) as well as the costs of movement with and without the help of the network to be constructed, are known. Further, the models take into consideration both the cost of everyday movement of the population and the cost of construction and maintenance of the network, the latter being determined by a given function on the total length of the network. The above data suffice, if one considers optimization in long-term prospective, while for the short-term optimization one also needs to know the transport plan of everyday movements of the population (i.e. the information on “who goes where”). Similar models can also be adapted for the optimization of networks of different nature, like telecommunication, pipeline or drainage networks. In the monograph we study the most general problem settings, namely, when neither the shape nor even the topology of the network to be constructed is a priori known.

To be more precise, given a region  $\Omega \subseteq \mathbb{R}^N$ , we will model the transportation network to be constructed by an a priori generic Borel set  $\Sigma \subseteq \Omega$ . We consider then the mass transportation problem in which the paths inside and outside the network  $\Sigma$  are charged differently. The aim is to find the best location for  $\Sigma$ , in order to minimize a suitable cost functional  $\mathfrak{F}(\Sigma)$ , which is given by the sum of the cost of transportation of the population, and the penalization term depending on the length of the network, which represents the cost of construction and maintenance of the network. To study the problem of existence of optimal solutions, we present first a relaxed version of the optimization problem, where the network is represented by a Borel measure rather than a set, and we prove the existence of a relaxed solution. We will study then the properties of optimal relaxed solutions (measures) and prove that, under suitable assumptions, the relaxed solution solve the original problem, i.e. in fact they correspond to rectifiable sets, and therefore can be called

classical solutions. However, it will be shown that in general the problem studied may have no classical solutions. We will also study some topological properties of optimal networks, like closedness and the number of connected components. In particular, we find rather sharp conditions on problem data, which ensure the existence of closed optimal networks and/or optimal networks having at most countably many connected components. Finally, we will prove a general regularity result on optimal networks. Namely, we will show that an optimal network is covered by a finite number of Lipschitz curves of uniformly bounded length, although it may have even uncountably many connected components.

# Acknowledgments

This work was conceived during the meeting *Giornate di Lavoro in “Calculus of Variations and Geometric Measure Theory”* held in Levico Terme (Italy), and was carried on thanks to the project “Calcolo delle Variazioni” (PRIN 2004) of the Italian Ministry of Education. The work of the third author was partially supported by the italian GNAMPA–INDAM.

# Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Problem Setting</b>	<b>7</b>
2.1	Notation and Preliminaries	7
2.2	Properties of Optimal Paths and Relaxed Costs	13
<b>3</b>	<b>Optimal Connected Networks</b>	<b>25</b>
3.1	Optimization Problem	25
3.2	Properties of the Optimal Networks	28
3.3	Average Distance Problem	33
<b>4</b>	<b>Relaxed Problem and Existence of Solutions</b>	<b>37</b>
4.1	Relaxed Problem Setting	37
4.2	Properties of Relaxed Minimizers	41
4.3	Non-existence of Classical Solutions	65
4.4	Existence of Classical Solutions	71
<b>5</b>	<b>Topological Properties of Optimal Sets</b>	<b>75</b>
5.1	Transiting Mass Function	75
5.2	Ordered Transport Path Measures	82
5.3	Closedness of Optimal Sets	92
5.4	Number of Connected Components of Optimal Sets	95
<b>6</b>	<b>Optimal Sets and Geodesics</b>	
	<b>in the Two-Dimensional Case</b>	<b>105</b>
6.1	Preliminary Constructions	106
6.2	Proof of the Main Result	120
	<b>Appendix</b>	<b>131</b>
A	The Mass Transportation Problem	131
B	Some Tools from Geometric Measure Theory	135
B.1	Measures as Duals of the Continuous Functions	135

B.2    Push-forward and Tensor Product of Measures ..... 140

B.3    Measure Valued Maps and Disintegration Theorem ... 140

B.4     $\Gamma$ –convergence ..... 142

**References** ..... 145

**Index** ..... 149