Lecture Notes in Mathematics 1947

FONDAZIONE CIME ROBERTO CONTI

CENTRO INTERNAZIONALE MATEMATICO ESTIVO
INTERNATIONAL MATHEMATICAL SUMMER CENTER

C.I.M.E. means Centro Internazionale Matematico Estivo, that is, International Mathematical Summer Center. Conceived in the early fifties, it was born in 1954 and made welcome by the world mathematical community where it remains in good health and spirit. Many mathematicians from all over the world have been involved in a way or another in C.I.M.E.'s activities during the past years.

So they already know what the C.I.M.E. is all about. For the benefit of future potential users and co-operators the main purposes and the functioning of the Centre may be summarized as follows: every year, during the summer, Sessions (three or four as a rule) on different themes from pure and applied mathematics are offered by application to mathematicians from all countries. Each session is generally based on three or four main courses (24−30 hours over a period of 6−8 working days) held from specialists of international renown, plus a certain number of seminars.

A C.I.M.E. Session, therefore, is neither a Symposium, nor just a School, but maybe a blend of both. The aim is that of bringing to the attention of younger researchers the origins, later developments, and perspectives of some branch of live mathematics.

The topics of the courses are generally of international resonance and the participation of the courses cover the expertise of different countries and continents. Such combination, gave an excellent opportunity to young participants to be acquainted with the most advance research in the topics of the courses and the possibility of an interchange with the world famous specialists. The full immersion atmosphere of the courses and the daily exchange among participants are a first building brick in the edifice of international collaboration in mathematical research.

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Enumerative Invariants in Algebraic Geometry and String Theory

Lectures given at the
C.I.M.E. Summer School
held in Cetraro, Italy
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Preface

Starting with the middle of the 1980s, there has been a growing and fruitful interaction between algebraic geometry and certain areas of theoretical high-energy physics, especially the various versions of string theory. In particular, physical heuristics have provided inspiration for new mathematical definitions (such as that of Gromov–Witten invariants) leading in turn to the solution of (sometimes classical) problems in enumerative geometry. Conversely, the availability of mathematically rigorous definitions and theorems has benefitted the physics research by providing needed evidence, in fields where experimental testing seems still very far in the future.

This process is still ongoing in the present day, and actually expanding. A partial reflection of it can be found in the courses of the CIME session *Enumerative invariants in algebraic geometry and string theory*. The session took place in Cetraro from June 6 to June 11, 2005 with the following courses:

- Dan Abramovich (Brown University): *Gromov–Witten Invariants for Orbifolds.* (5 h)
- Marcos Mariño (CERN): *Open Strings.* (5 h)
- Michael Thaddeus (Columbia University): *Moduli of Sheaves.* (5 h)
- Ravi Vakil (Stanford University): *Gromov–Witten Theory and the Moduli Space of Curves.* (5 h)

Moreover, the following two talks were given as complementary material to the course of Abramovich.

- Jim Bryan (University of British Columbia): *Quantum cohomology of orbifolds and their crepant resolutions.*
- Barbara Fantechi (Sissa): *Virtual fundamental class.*

Orbifolds are a natural generalization of complex manifolds, where local charts are given not by open subsets of a complex vector space but by their quotients by finite groups. There are two natural descriptions, one in terms of charts (which actually works in symplectic geometry) and one in algebraic

geometry as smooth complex Deligne–Mumford stacks. Physicists have long suggested treating orbifolds analogously to manifolds; this has led to the development of an orbifold Euler characteristic, then of orbifold Hodge numbers, and finally (due to Chen–Ruan, in 2000) of an orbifold cohomology, induced by a full theory of Gromov Witten invariants for orbifolds. The course of Dan Abramovich has presented the foundations, laid down in a series of papers by Abramovich, Vistoli and others, of the definition of Gromov–Witten invariants for orbifolds in the algebraic setting.

The course of Marcos Mariño presented explicit enumerative computations by manipulation of formal power series, based on the physical idea of transforming an open string theory to a closed string theory. The aim was to derive explicit relationships between Gromov–Witten, Donaldson–Thomas and Chern Simons invariants. In particular, the technique of the topological vertex was explained, which allows a cut-and-paste approach to the determination of such invariants. These methods have only partially received mathematical proofs; they are therefore an important source of conjectures and methods for further developments.

Enumerative geometry computations on moduli spaces of sheaves have long been extremely useful both in physics and in mathematics; for instance we may recall the definition of Donaldson and (more recently) Donaldson–Thomas invariants, for surfaces and (some) threefolds respectively. The course by Michael Thaddeus has been a very broad overview of this kind of techniques, with a particular accent on the definition of the Donaldson–Thomas invariants and the recent conjectures that relate them to Gromov–Witten invariants for Calabi Yau threefolds; evidence for the conjectures and examples illustrating their significance have also been included.

One of the more established parts of the algebraic geometry – high energy physics interaction has been the rigorous definition and the computation of Gromov–Witten invariants for smooth projective varieties. At the basis of the very definition there is the existence and properness of the moduli stack $\overline{M}_{g,n}$ of stable curves. Surprisingly, in recent years it has been possible to deduce theorems about $\overline{M}_{g,n}$ using the results of the Gromov–Witten theory. The course of Ravi Vakil gave a general introduction to this area of research, starting at a comparatively elementary level and then reaching proofs of some conjectures of C. Faber on the tautological cohomology ring of $\overline{M}_{g,n}$.

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