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Algebraic Groups and Lie Groups with Few Factors

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Preface

In the theory of locally compact topological groups, the aspects and notions from abstract group theory have conquered a meaningful place from the beginning (see New Bibliography in [44] and, e.g. [41–43]). Imposing group-theoretical conditions on the closed connected subgroups of a topological group has always been the way to develop the theory of locally compact groups along the lines of the theory of abstract groups.

Despite the fact that the class of algebraic groups has become a classical object in the mathematics of the last decades, most of the attention was concentrated on reductive algebraic groups. For an affine connected solvable algebraic group G , the theorem of Lie–Kolchin has been considered as definitive for the structure of G , whereas for connected non-affine groups, the attention turns to the analytic and homological aspects of these groups, which are quasi-projective varieties (cf. [79, 80, 89]). Complex Lie groups and algebraic groups as linear groups are an old theme of group theory, but connectedness of subgroups does not play a crucial rôle in this approach, as can be seen in [97]. Non-linear complex commutative Lie groups are a main subject of complex analysis (cf. [1, 7]).

In these notes we want to include systematically algebraic groups, as well as real and complex Lie groups, in the frame of our investigation. Although affine algebraic groups over fields of characteristic zero are related to linear Lie groups (cf. [11–13]), the theorems depending on the group topology differ (cf. e.g. Remark 5.3.6). For algebraic groups we want to stress the differences between algebraic groups over a field of characteristic $p > 0$ and over fields of characteristic zero.

One essential task of group theory is the description of a given group by its composition of more elementary groups. There are two kinds of most elementary Lie groups and algebraic groups. One class is formed by such groups that have a dense cyclic subgroup; such groups are commutative. In the class of locally compact groups these groups are determined in [15], in the class of algebraic groups (over a not necessarily algebraically closed field) they are classified in [24]. The other kind of elementary groups are those which have a chain as lattice of their subgroups. In the class of finite groups they are cyclic

groups of prime power order. In Lie groups and in algebraic groups the lattice of closed connected subgroups is a chain precisely in the following cases. If such groups do not have dimension one then in the class of Lie groups they are Shafarevich extensions of simple complex tori (cf. [7], Chapter 1, Section 6). In the class of algebraic groups over fields of characteristic zero they are either simple abelian varieties or extensions of a one-dimensional affine group by a simple abelian variety (cf. Theorem 4.1.3). In the class of algebraic groups over fields of positive characteristic the situation is much more complicated. Besides simple abelian varieties and extensions of a one-dimensional torus by a simple abelian variety there are also affine algebraic groups having a chain as lattice of connected closed subgroups. If they are commutative then they are Witt groups (cf. [89], Chapter 7, Sections 8 and 10), if they are not commutative then they form a very rich family of unipotent groups as our work shows.

Already J. Dieudonné was interested in groups having a chain as their lattice of closed connected subgroups; for such groups we introduce in this book the term *chain*. Namely, in [19], Section 7, he deals with non-commutative two-dimensional groups of this type and remarks that they are counter-examples to conjectures derived from the case of characteristic zero. In general, one reason for the importance of chains is the fact that a precise knowledge of them is indispensable for group theoretical investigations referring to the lattice of connected subgroups. Since the lattices of connected subgroups of the pre-image and the image of an algebraic or topological epimorphism with finite kernel are isomorphic (see [71], Lemma 1.3, p. 256), we consider groups related in this way as equivalent and use for them the term isogenous. More precisely, we use the notion of isogeny for algebraic and topological groups in the sense of [77], p. 417 (see also Section 2.1). This aspect motivated us to open, up to isogeny, the door to the exotic world of algebraic non-commutative chains which consists of unipotent chains since a reduction of algebraic chains to unipotent groups can be easily achieved. Even though over fields of positive characteristic not every connected algebraic group is generated by chains (see Remark 4.1.5), our work documents that they are the fundamental ingredients of unipotent groups over fields of positive characteristic.

Already the unipotent chains of nilpotency class two are difficult to treat. Namely, for a complete classification of them one needs a classification of all non-commutative extensions G of n -dimensional Witt groups by m -dimensional Witt groups such that G has an n -dimensional commutator subgroup. Despite these great obstacles, using the classification of two-dimensional unipotent groups given in [18], II, § 3, 4.6, p. 197, we could concretely determine (through hard and involved computations) all unipotent chains of dimension three over perfect fields of characteristic greater than two. In particular we obtain that any three-dimensional unipotent non-commutative chain over a perfect field of characteristic greater than two has nilpotency class two, its commutator subgroup has dimension one and the center is two-dimensional. An induction yields that for any unipotent chain

over a perfect field of characteristic greater than two, the commutator subgroup has co-dimension at least two and the center has dimension at least two (cf. Corollary 4.2.10). Moreover, the knowledge of three-dimensional chains allows us to classify, up to isogenies, all three-dimensional unipotent groups over perfect fields of characteristic greater than two (cf. Section 6). These results demonstrate the richness of examples in dimension three compared with dimension two. The plethora of unipotent k -groups over non-perfect fields and of dimension less or equal two (see [51]) justifies our restriction to algebraic groups over perfect fields.

Using the regular factor systems determining Witt groups as extensions of Witt groups by Witt groups (cf. [18], V, § 1, 1.4, p. 542 or [102]) and [18], II, § 3, 4.6, p. 197, we obtain a classification of unipotent chains over perfect fields of positive characteristic having a one-dimensional commutator subgroup (cf. Theorem 4.3.1).

As Remark 4.1.6 and Example 4.3.4 show, there are in any dimension unipotent chains of nilpotency class two having two-dimensional and three-dimensional commutator subgroup. But the involved structure of these extensions gives no hope that a complete classification of unipotent chains of nilpotency class two having a commutator subgroup of dimension greater than one could be achieved.

The classification of chains having a one-dimensional commutator subgroup yields a classification of connected algebraic groups G over perfect fields of characteristic $p > 2$ such that G has a central subgroup of co-dimension one. These groups have a representation as an almost direct product of a commutative group and a group which is a direct group of chains with amalgamated factor group (cf. Theorem 4.3.12). Moreover, we prove that in an algebraic group G having a central maximal connected subgroup the commutator subgroup G' is a (central) vector group. Conversely, if G' is a central vector group and $G/3G$ is isogenous to a Witt group, then the center $3G$ has co-dimension one in G (cf. Theorem 3.2.8). In contrast to this, a non-commutative algebraic group over a field of characteristic zero cannot have its center of co-dimension one.

Our investigations on chains G with one-dimensional commutator subgroup G' yield conditions under which an automorphism of the factor group G/G' can be extended to an automorphism of G . Using these results we can illustrate that the non-commutative chains are much more rigid than Witt groups. Namely, any connected algebraic group of algebraic automorphisms of a non-commutative unipotent chain of dimension greater than two is unipotent (cf. Corollary 4.3.11).

The lattice of normal connected algebraic subgroups of unipotent algebraic groups for which the nilpotency class is equal to their dimension n forms a chain of length n (cf. Proposition 3.1.13). Such unipotent groups occur only over fields of positive characteristic and play an opposite rôle to the one of chains which cannot have maximal nilpotency class (see Corollary 4.2.10). In Section 3.1 we introduce for any n a significant class of unipotent groups

$\mathfrak{J}_n(\alpha)$ of dimension n and nilpotency class n , characterise these groups as linear groups and study their structure. These groups show that a group with maximal nilpotency class can have a trivial adjoint representation (cf. Remark 3.1.6). At various places we use the groups $\mathfrak{J}_n(\alpha)$ as a source of counter-examples to find the limits of our theorems.

The nilpotency class of n -dimensional algebraic groups over fields of characteristic zero as well as n -dimensional real or complex Lie groups is at most $n - 1$. The Lie algebras corresponding to these groups of maximal nilpotency class are called *filiform Lie algebras* and form a class thoroughly studied for thirty years (cf. [36, 37]). The filiform groups, i.e. the groups having filiform Lie algebras, play in our results on groups in characteristic zero the same rôle as the unipotent chains in positive characteristic.

The simple structure of the lattice of connected subgroups of an algebraic or analytic chain motivated us to study to which extent individual properties of chains restrict the structure of algebraic and analytic groups. Most of these properties remain invariant under isogenies.

In Section 5.2 we investigate connected algebraic groups and connected Lie groups having exactly one maximal connected closed subgroup (*uni-maximal* groups) as well as connected algebraic groups and connected Lie groups having exactly one minimal connected closed subgroup (*uni-minimal* groups). The description of non-affine algebraic groups, respectively complex Lie groups, which are uni-minimal or uni-maximal easily reduces to extensions of affine groups of dimension at most one by abelian varieties, respectively to toroidal groups (cf. Theorem 5.2.7 and Proposition 5.2.4). Connected affine algebraic groups which are uni-minimal or uni-maximal and have dimension greater than one are unipotent algebraic groups over fields of positive characteristic (cf. Proposition 5.2.6).

A non-commutative connected unipotent algebraic group G is uni-maximal if and only if the commutator subgroup of every proper connected algebraic subgroup of G is smaller than the commutator subgroup of G (cf. Theorem 5.2.17). Any group in which the commutator subgroup is a maximal connected subgroup is uni-maximal; in particular the unipotent algebraic groups over fields of positive characteristic having maximal nilpotency class are of such type (cf. Section 3). But we construct in Remark 3.1.12 also numerous examples of uni-maximal algebraic groups in which the commutator subgroup is not maximal. Moreover, any non-commutative connected three-dimensional unipotent algebraic group over a field of positive characteristic which is not a product of two non-commutative chains is uni-maximal (see Theorem 6.4.5).

Uni-minimal non-commutative groups G turn out to be products of chains, where at most one factor C has dimension greater than two; if C is not commutative then the commutator subgroup of G coincides with the commutator subgroup of C (cf. Theorem 5.2.34). This result shows that the structure of uni-minimal groups is less complicated than the structure of uni-maximal groups.

In Corollary 5.2.35 we prove that the conditions to be uni-minimal and uni-maximal are strong enough to characterise the chains over fields of characteristic greater than two. Also the condition that in algebraic groups over fields of characteristic greater than two every proper algebraic subgroup is a chain characterises the chains up to two exceptions of small dimension (cf. Theorem 5.2.30). Moreover, a connected affine algebraic group over a field of arbitrary prime characteristic, containing a chain M as a maximal connected algebraic subgroup, is either a chain or a product of M with a chain of dimension at most two (cf. Theorem 5.2.40).

In chains with a one-dimensional commutator subgroup, any connected algebraic subgroup as well as any proper epimorphic image is commutative. In general however, for algebraic groups over fields of positive characteristic none of these two conditions is sufficient for a concrete description (cf. Corollary 5.2.23 and Proposition 5.2.38). In contrast to this, for real or complex Lie groups, for formal groups and for algebraic groups over fields of characteristic zero the assumption of commutativity of all proper connected subgroups as well as the dual condition of commutativity of all proper epimorphic images is strong enough for a classification. A powerful tool to achieve this goal is the classification of Lie algebras with one of these two properties. The Lie algebras in which every subalgebra is commutative have been studied thoroughly for thirty years (cf. [21, 30–32]). If G is a non-commutative connected affine algebraic group over a field of characteristic zero such that any connected algebraic subgroup is commutative, then G is at most three-dimensional (cf. Proposition 5.3.4). For formal groups we find an analogous situation (cf. Proposition 5.3.5). In contrast to this there exist real and complex Lie groups of any dimension having only commutative proper connected subgroups, they are precisely the extra-special real or complex Lie groups (see Remark 5.3.6). A connected non-simple non-commutative affine algebraic group of dimension at least three over a field of characteristic zero such that every epimorphic image of G is commutative is a Heisenberg group (cf. Corollary 5.3.9 and Proposition 5.3.11). A connected real or complex non-simple non-commutative Lie group of dimension greater than three having only commutative proper epimorphic images is an extra-special complex Lie group having as center a simple complex torus of dimension at least two.

An affine chain of dimension n has exactly one connected algebraic subgroup for any dimension $d \leq n$ and any two epimorphic images of the same dimension are isogenous. Investigating these two properties for connected algebraic groups, respectively for real or complex Lie groups, we call any such group *aligned* if any two proper connected closed subgroups of the same dimension are isomorphic, respectively *co-aligned* if all epimorphic images of the same dimension are isogenous. For algebraic groups over fields of positive characteristic the properties to be aligned and co-aligned are too weak to obtain a concrete description for such groups (see Section 5.7 and Theorem 6.4.9). Also for connected algebraic groups over fields k of characteristic zero and for real or complex Lie groups, the condition to be co-aligned alone is not strong

enough to obtain a reasonable description for groups having this property. In particular there is a rich family of nilpotent co-aligned algebraic k -groups as well as of nilpotent co-aligned real or complex Lie groups (see Remark 5.5.5). Only if we assume that these affine groups have nilpotency class two and k is algebraically closed we obtain Heisenberg groups (cf. Proposition 5.5.4).

In the class of solvable non-nilpotent connected affine algebraic groups G of dimension greater than three over algebraically closed fields of characteristic zero such that the unipotent radical of G is commutative, the co-aligned groups are precisely those for which the lattice of connected algebraic subgroups forms a projective geometry (see Theorem 5.5.16 and [71], Lemma 4.7, p. 262). If the unipotent radical U has nilpotency class two and G has dimension greater than four, then the property to be co-aligned characterises the semi-direct products of Heisenberg groups H with a one-dimensional torus T acting on H such that any closed connected subgroup of H is normalized but not centralised by T (see Theorem 5.5.21). However, the filiform groups admitting a non-trivial action of a one-dimensional torus show that a classification of co-aligned solvable non-nilpotent affine algebraic groups with unipotent radical of nilpotency class greater than two is not accessible. The same results hold for solvable non-nilpotent connected linear complex Lie groups of dimension greater than three, respectively four (cf. Theorem 5.5.16 and Theorem 5.5.21).

For co-aligned real Lie groups we meet the same difficulties as for algebraic groups and hence we arrive at a classification only for special subclasses. A connected non-commutative solvable real Lie group G of dimension greater than six having commutative commutator subgroup and containing non-trivial compact elements is co-aligned if and only if G is the direct product of a torus of dimension at most one and a semi-direct product of an even-dimensional vector group V with a one-dimensional torus such that any irreducible subspace of V has dimension two (see Proposition 5.5.23). A connected solvable real Lie group G such that the commutator subgroup G' is not commutative and the factor group G/G' has a non-trivial compact subgroup of dimension ≥ 2 is co-aligned if and only if G is a semi-direct product of two Heisenberg groups with amalgamated centre by a two-dimensional torus (cf. Proposition 5.5.27).

In contrast to the condition to be co-aligned, the property to be aligned is strong. This is documented by the fact that a non-commutative connected affine algebraic group over a field of characteristic zero or a linear complex Lie group is aligned if and only if it is unipotent and has dimension three (see Theorem 5.4.4 and Proposition 5.4.9). Moreover, the classification of the three-dimensional unipotent algebraic groups in Chapter 6 yields that a three-dimensional non-commutative connected unipotent algebraic group G over a perfect field of characteristic $p > 2$ which is aligned is uni-maximal (cf. Theorem 6.4.7). Furthermore, if G is an aligned uni-minimal group then G is a chain (cf. Corollary 6.4.8).

For non-linear complex Lie groups the condition to be aligned creates a situation which is more complicated (cf. Example 5.4.10). However, a

non-commutative connected real Lie group of dimension $n \geq 4$ is aligned if and only if it is locally isomorphic to one of the following compact Lie groups: $SO_2(\mathbb{R}) \times SO_3(\mathbb{R})$, $SO_3(\mathbb{R}) \times SO_3(\mathbb{R})$, $SU_3(\mathbb{C}, 0)$, $SO_5(\mathbb{R})$ and the 14-dimensional exceptional Lie group G_2 (cf. Theorem 5.4.8).

In Section 5.7 we characterise chains by the fact that they have only few non-isogenous factors. Namely a connected unipotent algebraic group is a chain if and only if it has only finitely many connected algebraic subgroups (cf. Theorem 5.7.1). This result allows far-reaching generalisations. For instance, a non-commutative unipotent algebraic group G is a chain if and only if every epimorphic image of G is isogenous to a subgroup of G and any two connected algebraic subgroups of G of the same dimension are isogenous (cf. Corollary 5.7.4). The dual conditions also give a characterisation of non-commutative unipotent chains (cf. Corollary 5.7.8).

In the theory of abstract groups, a group is called hamiltonian or sometimes a Dedekind group if all of its subgroups are normal. For algebraic groups this condition applied to all algebraic subgroups would not be interesting (see Theorem 7.2.1). Hence we say that a connected algebraic group is *hamiltonian* if all its connected algebraic subgroups are normal. For connected algebraic groups over a field of characteristic zero also this definition is too strong. Namely, we show that any connected algebraic k -group over a field k of characteristic zero such that any connected k -subgroup is normal is commutative (cf. Theorem 7.2.10). But the situation changes drastically if we consider hamiltonian groups over fields of positive characteristic. Any non-commutative chain, more generally any uni-minimal connected algebraic group, is hamiltonian (see Theorem 7.2.19). Other examples of connected hamiltonian algebraic groups are the groups in which the centre has co-dimension one. These groups in addition are *quasi-commutative*, i.e. algebraic groups where every commutative connected algebraic subgroup is central. We remark that non-commutative, but quasi-commutative algebraic groups exist only over fields of positive characteristic; they have nilpotency class two (Proposition 3.2.26).

Quite often replacing the condition of normality for certain subgroups by the condition of quasi-normality one obtains for abstract groups results of the same significance (see [93]). If G is an algebraic group and Q is a connected algebraic subgroup of G , then there are two natural possibilities to say that Q is quasi-normal. The stronger version is to demand that $QX = XQ$ for any algebraic subgroup of G . But with respect to this definition we can prove a sharper version of Theorem 1 in [87] : A connected algebraic k -subgroup P of a connected affine algebraic group G defined over an infinite perfect field k such that $PH = HP$ for any k -closed subgroup H of G is normal in G (see Theorem 7.1.1). Because of this result we call a connected algebraic subgroup Q of an algebraic group G *quasi-normal* if it is permutable with every connected algebraic subgroup of G . Quasi-normal, but not normal algebraic subgroups exist only in algebraic groups G over fields of positive characteristic (see Corollary 7.1.5). Essentially they are contained in the unipotent radical of G (see Corollary 7.1.8). Moreover, there are non-commutative algebraic groups

over fields of positive characteristic in which every algebraic subgroup is quasi-normal. Among these groups that we call *quasi-hamiltonian* there are groups which are not hamiltonian (e.g. Example 7.1.18 and Remark 7.1.20). A consequence of Corollary 7.1.8 is the fact that every connected quasi-hamiltonian algebraic group is nilpotent (cf. Theorem 7.1.11).

In Sections 7.1 and 7.2 we give many examples of quasi-hamiltonian and hamiltonian algebraic groups which are neither chains nor uni-minimal (see e.g. Example 7.2.18 and Remark 7.2.20). A big class of such groups is formed by the algebraic groups such that the factor group over their centers is a chain (see Theorem 7.1.12). Moreover, in these sections we describe some product constructions to obtain hamiltonian groups from given ones, e.g. from chains, and discuss which limitations occur.

A subclass of the class of hamiltonian algebraic groups are those groups in which every connected algebraic subgroup is characteristic; we call these groups *super-hamiltonian*. A classification of super-hamiltonian algebraic groups G is easy if G is a direct product of chains (see Corollaries 7.4.2, 7.4.3 and Proposition 7.4.4). However, the decision whether a product of chains is super-hamiltonian is difficult if the factors do not intersect trivially.

The experience from algebraic groups over fields of characteristic zero, from abelian varieties and from finite groups makes it surprising that over fields of prime characteristic there are many examples of three-dimensional non-commutative connected unipotent algebraic groups which are super-hamiltonian. In Section 7.4 we use our classification of three-dimensional connected unipotent groups over perfect fields of characteristic greater than two to decide which of these groups are super-hamiltonian. Although non-hamiltonian three-dimensional unipotent groups G exist if and only if the centre of G is one-dimensional, super-hamiltonian three-dimensional unipotent groups G do not exist only under severe restrictions on the centre, the commutator subgroup G' and the factor group G/G' (see Propositions 7.2.6 and 7.4.6).

Connected quasi-normal subgroups of connected affine algebraic groups over fields of characteristic zero, respectively of connected Lie groups, are treated in [87], respectively [86]. A connected closed subgroup Q of a topological group G is defined to be quasi-normal if it is *topologically permutable* with any closed subgroup of G , i.e. if the sets QP and PQ have the same closure for any closed subgroup P of G (see [52]). This fact motivated us to seek in Section 7.3 a unified method for the study of quasi-normal subgroups in topological and algebraic groups G . It turns out that a unified treatment is possible using a suitable closure operator on the set of subgroups of G (cf. Definition 7.3.4). This procedure allows to prove that in a connected real or complex Lie group G , any connected closed subgroup which is topologically permutable with every closed connected subgroup of G must be normal in G (see Theorem 7.3.5). In the case of real or complex Lie groups this is a positive answer to a conjecture in [53]. Moreover, as a consequence we obtain that every connected real or complex Lie group in which every connected closed

subgroup is topologically permutable with any other connected closed subgroup must be commutative (see Corollary 7.3.7). Our point of view has also the advantage that p -adic Lie groups are included in the considerations, provided we modify topological permutability to locally topological permutability (cf. Definition 7.3.10). Using this we can for instance show that any p -adic Lie group G (over an ultrametric field of characteristic zero) contains an open commutative subgroup if any family of subgroups which corresponds to some subalgebra in the Lie algebra of G is locally permutable with every other such family (cf. Theorem 7.3.14).

In our work we extend all results about affine algebraic groups, respectively about linear complex Lie groups, to non-affine algebraic groups, respectively to non-linear complex Lie groups. To do this we use for algebraic groups a series of well-known results of M. Rosenlicht (cf. Section 1.3) and for complex Lie groups some theorems on complex tori and toroidal groups (cf. Section 1.1). Moreover, for algebraic groups we discuss rationality questions and try to generalize our results to algebraic k -groups.

In Chapter 1 we collect known results on real and complex Lie groups, formal groups, p -adic groups and algebraic groups as far as they are needed for our investigations.

The main part of Section 2.1 is devoted to the theory of extensions of algebraic groups, in view of our use of regular factor systems. In particular, we need to know sufficient conditions for an algebraic k -group G , which is an extension of an algebraic k -group A by an algebraic k -group B , to be k -isomorphic to a k -group defined on the k -variety $B \times A$. Although such a representation of G is not always possible (see e.g. Remarks 2.1.4 and 2.1.5), it exists if the normal subgroup A is k -split and unipotent and B is affine (cf. [82], Theorem 1, p. 99). Another aim of Section 2.1 is to discuss relations between factor systems and isogenies. If there exists an isogeny from a connected commutative unipotent group G_1 onto G_2 , then an isogeny from G_2 onto G_1 exists as well (see [89], Proposition 10, p. 176). One could ask if this holds for *unipotent* connected non-commutative algebraic groups. Example 2.1.14 answers this question negatively. We establish in Proposition 2.1.12 necessary and sufficient conditions for the existence of an isogeny between central extensions of A_1 by B_1 and A_2 by B_2 which extends two given isogenies between A_1 and A_2 and between B_1 and B_2 . In Proposition 2.1.13 we give sufficient conditions for central extensions of a Witt group or of a vector group by an algebraic group to be isogenous in sense of [77], p. 417 (see also Section 2.1). Moreover, for two-dimensional non-commutative unipotent groups over perfect fields of characteristic $p > 2$ and of exponent p we give another procedure to obtain, up to a coboundary, all other factor systems from a suitable one (see Proposition 2.1.9).

In Section 2.2 we deal with commutative extensions of commutative connected complex Lie groups. Since any such group is (holomorphically) isomorphic to the direct product of a linear torus $(\mathbb{C}^*)^m$, a vector group \mathbb{C}^l and a toroidal group X , the theory of commutative extensions of commutative connected complex Lie groups reduces to the case of extensions which are

toroidal groups. The complex n -dimensional toroidal groups X are completely described by means of period matrices, the columns of which are vectors of the lattice Λ determining X . If the \mathbb{R} -span of the lattice Λ has dimension $2n$, then X is a compact group, namely a complex torus. The description of complex tori by period matrices given in [7], Chapter 1, suggests to us a generalization to toroidal groups. Using period matrices, necessary and sufficient conditions for two toroidal groups to be isogenous, respectively isomorphic, are obtained. Propositions 2.2.2 and 2.2.3 allow us to recognize suitable closed subgroups and factor groups of a toroidal group within the corresponding period matrix. Moreover, a non-compact toroidal group can contain more than one maximal closed complex linear torus (cf. Example 2.2.4). Using these results we can concretely decide under which circumstances commutative holomorphic extensions of toroidal groups by toroidal groups split, and we show that, analogously to the case of complex tori, Shafarevich extensions of toroidal groups by toroidal groups exist.

If one considers, as for connected algebraic groups, connected real and complex Lie groups in which every connected closed subgroup is characteristic, then the Shafarevich extensions are prominent examples of super-hamiltonian Lie groups. If a super-hamiltonian Lie group has dimension greater than two, then it is a commutative complex Lie group which is a closed subgroup of the direct product having as factors a one-dimensional vector group, a one-dimensional linear torus and a non-trivial toroidal group which is super-hamiltonian. To settle the question when in a toroidal group X any connected closed subgroup is characteristic is easy if X is the direct product of Shafarevich extensions of a simple torus by a simple torus. In this case X is super-hamiltonian if and only if there is no non-trivial homomorphism between two distinct direct factors of X . In general, however, only a thorough analysis of the period matrix determining X allows to decide whether X is super-hamiltonian or not (see Proposition 7.4.11, Example 7.4.12 and Proposition 7.4.13).

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