

Lecture Notes in Mathematics

1942

Editors:

J.-M. Morel, Cachan

F. Takens, Groningen

B. Teissier, Paris



**FONDAZIONE
CIME**
ROBERTO CONTI
CENTRO INTERNAZIONALE MATEMATICO ESTIVO
INTERNATIONAL MATHEMATICAL SUMMER CENTER

C.I.M.E. means Centro Internazionale Matematico Estivo, that is, International Mathematical Summer Center. Conceived in the early fifties, it was born in 1954 and made welcome by the world mathematical community where it remains in good health and spirit. Many mathematicians from all over the world have been involved in a way or another in C.I.M.E.'s activities during the past years.

So they already know what the C.I.M.E. is all about. For the benefit of future potential users and co-operators the main purposes and the functioning of the Centre may be summarized as follows: every year, during the summer, Sessions (three or four as a rule) on different themes from pure and applied mathematics are offered by application to mathematicians from all countries. Each session is generally based on three or four main courses (24–30 hours over a period of 6-8 working days) held from specialists of international renown, plus a certain number of seminars.

A C.I.M.E. Session, therefore, is neither a Symposium, nor just a School, but maybe a blend of both. The aim is that of bringing to the attention of younger researchers the origins, later developments, and perspectives of some branch of live mathematics.

The topics of the courses are generally of international resonance and the participation of the courses cover the expertise of different countries and continents. Such combination, gave an excellent opportunity to young participants to be acquainted with the most advance research in the topics of the courses and the possibility of an interchange with the world famous specialists. The full immersion atmosphere of the courses and the daily exchange among participants are a first building brick in the edifice of international collaboration in mathematical research.

C.I.M.E. Director	C.I.M.E. Secretary
Pietro ZECCA	Elvira MASCOLO
Dipartimento di Energetica "S. Stecco"	Dipartimento di Matematica
Università di Firenze	Università di Firenze
Via S. Marta, 3	viale G.B. Morgagni 67/A
50139 Florence	50134 Florence
Italy	Italy
e-mail: zecca@unifi.it	e-mail: mascolo@math.unifi.it

For more information see CIME's homepage: <http://www.cime.unifi.it>

CIME's activity is supported by:

- Istituto Nazionale di Alta Matematica "F. Severi"
- Ministero degli Affari Esteri - Direzione Generale per la Promozione e la Cooperazione - Ufficio V
- Ministero dell'Istruzione, dell'Università e delle Ricerche

Sergio Albeverio · Franco Flandoli
Yakov G. Sinai

SPDE in Hydrodynamic: Recent Progress and Prospects

Lectures given at the
C.I.M.E. Summer School
held in Cetraro, Italy
August 29–September 3, 2005

Editors:
Giuseppe Da Prato
Michael Röckner

 Springer



Sergio Albeverio
Institut für Angewandte Mathematik
Universität Bonn
Wegelerstr. 6
53115 Bonn
Germany
albeverio@uni-bonn.de

Franco Flandoli
Dipartimento di Matematica Applicata
“U.Dini”
Università di Pisa
Via Buonarroti 1
56127 Pisa
Italy
flandoli@dma.unipi.it

Yakov G. Sinai
Department of Mathematics
Princeton University
Fine Hall, Washington Road
Princeton N.J. 08544
USA
sinai@math.princeton.edu

Giuseppe Da Prato
Scuola Normale Superiore
Piazza dei Cavalieri 7
56126 Pisa
Italy
daprato@sns.it

Michael Röckner
Fakultät für Mathematik
Universität Bielefeld
Postfach 100131
33501 Bielefeld
Germany
roeckner@math.uni-bielefeld.de

ISBN: 978-3-540-78492-0 e-ISBN: 978-3-540-78493-7
DOI: 10.1007/978-3-540-78493-7

Lecture Notes in Mathematics ISSN print edition: 0075-8434
ISSN electronic edition: 1617-9692

Library of Congress Control Number: 2008921567

Mathematics Subject Classification (2000): 76D03, 35Q05, 76F02, 60H15

© 2008 Springer-Verlag Berlin Heidelberg

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilm or in any other way, and storage in data banks. Duplication of this publication or parts thereof is permitted only under the provisions of the German Copyright Law of September 9, 1965, in its current version, and permission for use must always be obtained from Springer. Violations are liable to prosecution under the German Copyright Law.

The use of general descriptive names, registered names, trademarks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

Cover design: WMXDesign GmbH

Cover art: The Togliatti Quintic by Oliver Labs, produced with Stephan Endrass' visualization tool surf (<http://surf.sourceforge.net>)

Printed on acid-free paper

9 8 7 6 5 4 3 2 1

springer.com

Preface

It is a commonly accepted fact in the mathematical scientific community that the rigorous understanding of turbulence and related questions in hydrodynamics is one of the most important problems in mathematics and one of the challenging tasks for the future development of the theory of partial differential equations in particular, but also of analysis in general. One of the central open problems, namely the well posedness of the 3D-Navier-Stokes system has been selected as one of the millennium problems and has resisted all attempts to solve it up to the present day.

Over more than half a century a lot of deep mathematics was developed to tackle these problems. One of the approaches was to use stochastic analysis based on modifying the equations (as e.g. Euler, Navier-Stokes and Burgers) adding a noise term. The idea here was to use the smoothing effect of the noise on the one hand, but also to discover new phenomena of stochastic nature on the other hand. In addition, this was also motivated by physical considerations, aiming at including perturbative effects, which cannot be modelled deterministically, due to too many degrees of freedom being involved, or aiming at taking into account different time scales of the components of the underlying dynamics. Today we look back on 30 years of mathematical work implementing probabilistic ideas into the area. During the last few years activities have become even more intense and several new groups in the world working on probability have turned their attention to these classes of highly interesting SPDEs of fundamental importance in Physics.

In a sentence, one of the purposes of the course was to understand the link between the Euler and Navier-Stokes equations or their stochastic versions and the phenomenological laws of turbulence. The idea can be better understood by analogy with Feynmans description of statistical mechanics: in that field on the one hand one has the Hamilton (or Schrödinger) equations for the dynamics of molecules and, on the other hand, the macroscopic laws of thermodynamics. In between there is the concept of Gibbs measures, so the theory looks like the ascent of a mountain, from Hamilton equations to Gibbs measures (here ergodicity is a central topic), and a descent from Gibbs measures

to thermodynamics. Translating this viewpoint in the realm of turbulent fluid dynamics, on the one side we have the Navier–Stokes equations as dynamical equations (that we commonly accept as essentially correct). On the other side, we know a number of phenomenological laws, like the Kolmogorov’s scaling (which he proposed in 1941, therefore called K41 scaling) or the multifractal scalings, which fit experimental and numerical data to some extent, but miss a rigorous foundation and presumably require some correction. If we aim at an analogy with statistical mechanics, the missing point is a concept of Gibbs measure linking these two extreme parts of the theory.

Three courses of eight hours each were delivered to develop these ideas, both for the deterministic and the stochastic case.

Sergio Albeverio presented an approach to (deterministic) Euler and stochastic Navier–Stokes equations in two dimensions based on invariant measures and renormalization methods. His last lecture was devoted to asymptotic methods for functional integrals.

Franco Flandoli started from some basic results on Navier–Stokes equations in three dimensions, discussing topics as existence of martingale solutions, construction of a transition semigroup, ergodicity and continuous dependence on initial conditions. One of the main results was a preliminary step to prove well posedness of the stochastic 3D-Navier–Stokes equations by showing the existence of a Markov selection. He also has presented a review of the Kolmogorov K41 scaling law and some rigorous results on it for the stochastic Navier–Stokes equations.

Finally, Yakov Sinai described some rigorous mathematical results for d -dimensional (deterministic) Navier–Stokes systems. In this direction he explained the power series and diagrams method for the Fourier transform of Navier–Stokes equations and the Foias-Temam Theorem. He also presented some recent results on the one-dimensional Burgers equation with random forcing, that is, in the stochastic case.

Afternoon sessions were devoted to research seminars delivered by the participants.

We thank the lecturers and all participants for their contributions to the success of this Summer School.

Finally, we thank the CIME Scientific Committee for giving us the opportunity to organize this meeting and the CIME staff for their efficient and continuous help.

Contents

Some Methods of Infinite Dimensional Analysis in Hydrodynamics: An Introduction

<i>Sergio Albeverio and Benedetta Ferrario</i>	1
1 Introduction	1
2 The Euler Equation, its Invariants and Associated Invariant Measures	2
2.1 The Euler Equation	2
2.2 The Euler Equation in Terms of Vorticity	3
2.3 The Conserved Quantities for the Euler Equation	4
2.4 Heuristic Invariant Measures Associated with the Euler Equation	6
2.5 The Euler Equation in Terms of the Fourier Components of the Stream Function	8
2.6 The Necessity of Looking for Singular Solutions. Divergence of the Energy with Respect to the Enstrophy Measure	9
2.7 Relation of the Enstrophy Measure μ_γ with the Euler Equation	10
2.8 The Infinitesimal Invariance of μ_γ . Relation with the “Hopf Approach”	11
2.9 The Question of Uniqueness of Generators of the Euler Flow	12
2.10 An Euler Flow in a Sobolev Space of Negative Index	14
2.11 Some Remarks on the Vortex Model and its Relations with the Euler Equation	14
3 Stochastic Navier–Stokes Equation	16
3.1 The Navier–Stokes Equation with Space-Time White Noise	17
3.2 The Gaussian Invariant Measure Given by the Enstrophy (and the Viscosity Parameter)	20
3.3 Existence of Strong Solutions	23
3.4 Pathwise Uniqueness	29
3.5 Some Additional Remarks and Complements	33
3.6 Appendix	35
References	36

An Introduction to 3D Stochastic Fluid Dynamics

<i>Franco Flandoli</i>	51
1 Introduction	51
2 Abstract Framework and General Preliminaries	52
3 Finite Dimensional Models	57
3.1 Introduction and Examples	57
3.2 A Priori Bounds	60
3.3 Comparison of Two Solutions and Pathwise Estimates	71
3.4 Existence and Uniqueness, Markov Property	73
3.5 Invariant Measures	75
3.6 Galerkin Stationary Measures for the 3D Equation	79
4 Stochastic Navier–Stokes Equations in 3D	82
4.1 Concepts of Solution	82
4.2 Existence of Solutions to the Martingale Problem	86
4.3 Technical Complements	91
4.4 An Abstract Markov Selection Result	95
4.5 Markov Selection for the 3D Stochastic NSE’s	103
4.6 Continuity in u_0 of Markov Solutions	109
5 Some Topics on Turbulence	124
5.1 Introduction and a Few Keywords	124
5.2 K41 Scaling Law: Heuristics and Unclear Issues	126
5.3 Definitions and Examples	129
5.4 Brownian Eddies and Random Vortex Filaments	133
5.5 Necessary Conditions for K41	139
5.6 A Condition Equivalent to K41	144
References	148

Mathematical Results Related to the Navier–Stokes System

<i>Yakov G. Sinai</i>	151
1 Introduction	151
2 Power Series and Diagrams for the Navier–Stokes-System	154
3 Foias-Temam Theorem for $2D$ –Navier–Stokes System with Periodic Boundary Condition	158
4 Burgers System and $1 - D$ Inviscid Burgers Equation with Random Forcing	161
References	163

List of Participants	165
-----------------------------------	-----