Lecture Notes in Mathematics 1942

FONDAZIONE CIME ROBERTO CONTI
CENTRO INTERNAZIONALE MATEMATICO ESTIVO
INTERNATIONAL MATHEMATICAL SUMMER CENTER

C.I.M.E. means Centro Internazionale Matematico Estivo, that is, International Mathematical Summer Center. Conceived in the early fifties, it was born in 1954 and made welcome by the world mathematical community where it remains in good health and spirit. Many mathematicians from all over the world have been involved in a way or another in C.I.M.E.'s activities during the past years.

So they already know what the C.I.M.E. is all about. For the benefit of future potential users and co-operators the main purposes and the functioning of the Centre may be summarized as follows: every year, during the summer, Sessions (three or four as a rule) on different themes from pure and applied mathematics are offered by application to mathematicians from all countries. Each session is generally based on three or four main courses (24–30 hours over a period of 6-8 working days) held from specialists of international renown, plus a certain number of seminars.

A C.I.M.E. Session, therefore, is neither a Symposium, nor just a School, but maybe a blend of both. The aim is that of bringing to the attention of younger researchers the origins, later developments, and perspectives of some branch of live mathematics.

The topics of the courses are generally of international resonance and the participation of the courses cover the expertise of different countries and continents. Such combination, gave an excellent opportunity to young participants to be acquainted with the most advance research in the topics of the courses and the possibility of an interchange with the world famous specialists. The full immersion atmosphere of the courses and the daily exchange among participants are a first building brick in the edifice of international collaboration in mathematical research.

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SPDE in Hydrodynamic: Recent Progress and Prospects

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Preface

It is a commonly accepted fact in the mathematical scientific community that the rigorous understanding of turbulence and related questions in hydrodynamics is one of the most important problems in mathematics and one of the challenging tasks for the future development of the theory of partial differential equations in particular, but also of analysis in general. One of the central open problems, namely the well posedness of the 3D-Navier–Stokes system has been selected as one of the millennium problems and has resisted all attempts to solve it up to the present day.

Over more than half a century a lot of deep mathematics was developed to tackle these problems. One of the approaches was to use stochastic analysis based on modifying the equations (as e.g. Euler, Navier–Stokes and Burgers) adding a noise term. The idea here was to use the smoothing effect of the noise on the one hand, but also to discover new phenomena of stochastic nature on the other hand. In addition, this was also motivated by physical considerations, aiming at including perturbative effects, which cannot be modelled deterministically, due to too many degrees of freedom being involved, or aiming at taking into account different time scales of the components of the underlying dynamics. Today we look back on 30 years of mathematical work implementing probabilistic ideas into the area. During the last few years activities have become even more intense and several new groups in the world working on probability have turned their attention to these classes of highly interesting SPDEs of fundamental importance in Physics.

In a sentence, one of the purposes of the course was to understand the link between the Euler and Navier–Stokes equations or their stochastic versions and the phenomenological laws of turbulence. The idea can be better understood by analogy with Feynmans description of statistical mechanics: in that field on the one hand one has the Hamilton (or Schrödinger) equations for the dynamics of molecules and, on the other hand, the macroscopic laws of thermodynamics. In between there is the concept of Gibbs measures, so the theory looks like the ascent of a mountain, from Hamilton equations to Gibbs measures (here ergodicity is a central topic), and a descent from Gibbs measures

to thermodynamics. Translating this viewpoint in the realm of turbulent fluid dynamics, on the one side we have the Navier–Stokes equations as dynamical equations (that we commonly accept as essentially correct). On the other side, we know a number of phenomenological laws, like the Kolmogorov's scaling (which he proposed in 1941, therefore called K41 scaling) or the multifractal scalings, which fit experimental and numerical data to some extent, but miss a rigorous foundation and presumably require some correction. If we aim at an analogy with statistical mechanics, the missing point is a concept of Gibbs measure linking these two extreme parts of the theory.

Three courses of eight hours each were delivered to develop these ideas, both for the deterministic and the stochastic case.

Sergio Albeverio presented an approach to (deterministic) Euler and stochastic Navier–Stokes equations in two dimensions based on invariant measures and renormalization methods. His last lecture was devoted to asymptotic methods for functional integrals.

Franco Flandoli started from some basic results on Navier–Stokes equations in three dimensions, discussing topics as existence of martingale solutions, construction of a transition semigroup, ergodicity and continuous dependence on initial conditions. One of the main results was a preliminary step to prove well posedness of the stochastic 3D-Navier–Stokes equations by showing the existence of a Markov selection. He also has presented a review of the Kolmogorov K41 scaling law and some rigorous results on it for the stochastic Navier–Stokes equations.

Finally, Yakov Sinai described some rigorous mathematical results for d-dimensional (determinisitic) Navier–Stokes systems. In this direction he explained the power series and diagrams method for the Fourier transform of Navier–Stokes equations and the Foias-Temam Theorem. He also presented some recent results on the one-dimensional Burgers equation with random forcing, that is, in the stochastic case.

Afternoon sessions were devoted to research seminars delivered by the participants.

We thank the lecturers and all participants for their contributions to the success of this Summer School.

Finally, we thank the CIME Scientific Committee for giving us the opportunity to organize this meeting and the CIME staff for their efficient and continuous help.

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Giuseppe Da Prato
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