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Arithmetical Investigations

Representation Theory, Orthogonal
Polynomials, and Quantum Interpolations

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ISBN 978-3-540-78378-7 e-ISBN 978-3-540-78379-4
DOI: 10.1007/978-3-540-78379-4

Lecture Notes in Mathematics ISSN print edition: 0075-8434
ISSN electronic edition: 1617-9692

Library of Congress Control Number: 2008921367

Mathematics Subject Classification (2000): 11-02, 11S80, 11S85

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Cover design: WMXDesign GmbH

Printed on acid-free paper

9 8 7 6 5 4 3 2 1

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To Yedidya, Antonia, Elisha, Yehonadav,
Amiad & Yoad.

Preface

This book grew out of lectures given at Kyushu University under the support of the Twenty-first Century COE Program “Development of Dynamical Mathematics with High Functionality” (Program Leader: Prof. Mitsuhiro Nakao). They were meant to serve as a primer to my book [Har5]. Indeed that book is very condense, and hard to read. We included however many new themes, such as the higher rank generalization of [Har5], and the fundamental semi-group. Since the audience consisted mainly of representation theorists, the focus shifted more into representation theory (hence less into geometry). We kept the lecture flair, sometimes explaining basic material in more detail, and sometimes only giving brief descriptions.

This book would have never come to life without the many efforts of Professor Masato Wakayama. The author thanks him also for his incredible hospitality. Thanks are also due to Yoshinori Yamasaki, who did an excellent job of writing down and typing the lectures into \LaTeX .

July 2006

Haifa

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