Lecture Notes in Mathematics 1938

C.I.M.E. means Centro Internazionale Matematico Estivo, that is, International Mathematical Summer Center. Conceived in the early fifties, it was born in 1954 and made welcome by the world mathematical community where it remains in good health and spirit. Many mathematicians from all over the world have been involved in a way or another in C.I.M.E.'s activities during the past years.

So they already know what the C.I.M.E. is all about. For the benefit of future potential users and co-operators the main purposes and the functioning of the Centre may be summarized as follows: every year, during the summer, Sessions (three or four as a rule) on different themes from pure and applied mathematics are offered by application to mathematicians from all countries. Each session is generally based on three or four main courses (24–30 hours over a period of 6-8 working days) held from specialists of international renown, plus a certain number of seminars.

A C.I.M.E. Session, therefore, is neither a Symposium, nor just a School, but maybe a blend of both. The aim is that of bringing to the attention of younger researchers the origins, later developments, and perspectives of some branch of live mathematics.

The topics of the courses are generally of international resonance and the participation of the courses cover the expertise of different countries and continents. Such combination, gave an excellent opportunity to young participants to be acquainted with the most advance research in the topics of the courses and the possibility of an interchange with the world famous specialists. The full immersion atmosphere of the courses and the daily exchange among participants are a first building brick in the edifice of international collaboration in mathematical research.

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CIME's activity is supported by:

– Istituto Nationale di Alta Mathematica "F. Severi"
– Ministero dell'Istruzione, dell'Università e delle Ricerca
– Ministero degli Affari Esteri, Direzione Generale per la Promozione e la Cooperazione, Ufficio V

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Symplectic 4-Manifolds and Algebraic Surfaces

Lectures given at the
C.I.M.E. Summer School
held in Cetraro, Italy
September 2–10, 2003

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Springer

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CIME
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ISBN: 978-3-540-78278-0 e-ISBN: 978-3-540-78279-7
DOI: 10.1007/978-3-540-78279-7

Lecture Notes in Mathematics ISSN print edition: 0075-8434
ISSN electronic edition: 1617-9692

Library of Congress Control Number: 2008921369

Mathematics Subject Classification (2000): 14J29, 14D66, 53D99, 32G05, 32G15, 20F36, 37R50

Cover design: *design & production* GmbH, Heidelberg

Cover art: The Togliatti Quintic by Oliver Labs, produced with Stephan Endrass' visualization tool surf (http://surf.sourceforge.net)

Printed on acid-free paper

9 8 7 6 5 4 3 2 1

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Preface

The third C.I.M.E. Session 'Symplectic 4-Manifolds and Algebraic Surfaces' took place from September 2 to September 10, 2003 in the customary beautiful location of the Grand Hotel San Michele, Cetraro, Cosenza.

The present volume contains the text of the five series of lectures, which were delivered during the course.

There were also some very interesting seminar lectures during the course. They are of a more specialized nature and are not reproduced here.

The lectures survey recent and important advances in symplectic and differential topology of 4-manifolds and algebraic surfaces.

Relations with real algebraic geometry have been treated only in part in the course by Catanese, and much more in the seminars by Frediani and Welschinger. Indeed, this and other very interesting topics of current research could only be treated rather quickly, in view of the vastness of the central theme of the School, the study of differential, symplectic and complex structures on even dimensional, and especially four-dimensional, manifolds.

The course had at least a double valency: on the one hand the introduction of new methods, for instance symplectic geometry, for the study of moduli spaces of complex structures on algebraic surfaces. On the other hand, the use of algebraic surfaces as concrete models for the investigation of symplectic topology in dimension 4, and for laying down a research plan based on the analogies with the surface classification.

One concrete example of the synergy of these two viewpoints is given for instance by the study of partial compactifications of moduli spaces of singular surfaces, which led to the construction of symplectomorphisms through surgeries associated with smoothings of singularities.

Let us now try to describe the contents of the courses and the interwoven thread which relates them to one another, thus making this volume a coherent exposition of an active field of current research.

As it is well known, every projective variety inherits from the ambient space the Fubini-Study Kähler form, and in this way one obtains the most natural examples of symplectic structures. Even if one wants to consider more

general symplectic manifolds, for use in the theory of dynamical systems, or just for the sake of classification, the relation with complex manifolds theory is always present.

In fact, a symplectic manifold always admits almost-complex structures, and the Kähler condition has as an analogue the condition that the almost-complex structure be compatible with the given symplectic structure. Even if almost-complex structures are not integrable (i.e., there are no local holomorphic coordinates), nevertheless one can still consider maps from a complex curve to the given almost-complex manifold whose derivative is complex linear.

One of the fundamental ideas, due to Gromov, is to use such maps to study the topology of symplectic manifolds.

These maps are called pseudoholomorphic curves, and the key point is that the corresponding generalized Cauchy–Riemann equations in the nonintegrable case do not substantially differ from the classical case (analyticity, removable singularities,...). The great advantage is, however, that while complex structures may remain 'nongeneric' even after deformation, a generic almost-complex structure really has an apt behaviour for transversality questions.

The study of pseudoholomorphic curves leads to important invariants, the so-called Gromov–Witten invariants, which have become an active research subject since late 1993 and were also treated in the seminar by Pandharipande.

If we start from a complex manifold, there remains, however, a basic question: if we take symplectic curves, how much do these differ from the holomorphic curves pertaining to the initial complex structure? Can they be deformed isotopically to holomorphic curves? Its analogue in algebraic geometry is to classify holomorphic curves under algebraic equivalence. In general, there are complex manifolds which contain symplectic curves, not isotopic to holomorphic ones. However, if the underlying symplectic manifold is an algebraic surface with positive first Chern class, it is expected that any symplectic curve be isotopic to a holomorphic curve.

This is the so-called symplectic isotopy problem, one of the fundamental problems in the study of symplectic 4-manifolds. It has many topological consequences in symplectic geometry. For instance, the solution to this symplectic isotopy problem provides a very effective way of classifying simply-connected symplectic 4-manifolds. Significant progress has been made. The central topic of the course by Siebert and Tian was the study of the symplectic isotopy problem, describing main tools and recent progress. The course has a pretty strong analytic flavour because of the nonlinearity of the Cauchy–Riemann equations in the nonintegrable case. Some applications to symplectic 4-manifolds were also discussed, in particular, that any genus two symplectic Lefschetz fibration under some mild non-degeneracy conditions is equivalent to a holomorphic surface. This result ties in with what we might call the 'dual' approach.

This is the approach taken by Donaldson for the study of symplectic 4-manifolds. Donaldson was able to extend the algebro-geometric concept of Lefschetz pencils to the case of symplectic manifolds. Even if for any generic almost-complex structure one cannot find holomorphic functions (even

locally); one can nevertheless find smooth functions and sections of line bundles whose antiholomorphic derivative ($\bar{\partial} f$), even if not zero, is still much smaller than the holomorphic derivative ∂f. This condition produces the same type of topological behaviour as the one possessed by holomorphic functions, and the functions satisfying it are called approximately holomorphic functions (respectively, sections).

In this way, Donaldson was able to extend the algebro-geometric concept of Lefschetz pencils to the case of symplectic manifolds.

The topic of Lefschetz pencils has occupied a central role in several courses, one by Auroux and Smith, one by Seidel, and one by Catanese.

In fact, one main use of Lefschetz pencils is, from the results of Kas and Gompf, to encode the differential topology and the symplectic topology of the fibred manifold into a factorization inside the mapping class group, with factors which are (positive) Dehn twists.

While the course of Auroux and Smith used symplectic Lefschetz pencils to study topological invariants of symplectic 4-manifolds and the differences between the world of symplectic 4-manifolds and that of complex surfaces, in the course by Catanese, Lefschetz fibrations were used to describe recent work done in collaboration with Wajnryb to prove explicit diffeomorphisms of certain simply connected algebraic surfaces which are not deformation equivalent.

An important ingredient here is a detailed knowledge of the mapping class group of the fibres, which are compact complex curves of genus at least two.

Applications to higher dimensional symplectic varieties, and to Mirror symmetry, were discussed in the course by Seidel, dedicated to the symplectic mapping class group of 4-manifolds, and to Dehn twists in dimension 4. The resulting picture of symplectic monodromy is surprising. In fact, Seidel shows that the natural homomorphism of the symplectic to the differential mapping class group may not be injective, and moreover reveals a delicate deformation behaviour: there are symplectomorphisms which are not isotopic to the identity for some special symplectic structure, but become isotopic after a small deformation of the symplectic structure.

Lefschetz pencils are the ‘generic’ maps with target the complex projective line: the course by Auroux and Smith went all the way to consider a generalization of the classical algebro-geometric concept of ‘generic multiple planes’.

This notion was extended to the symplectic case, again via approximately holomorphic sections, by Auroux and Katzarkov, and the course discusses the geometric invariants of a symplectic structure (which are deformation invariant) that can be extracted in this way.

This research interest is related to an old problem, posed by Boris Moishezon, namely, whether one can distinguish connected components of moduli spaces of surfaces of general type via these invariants (essentially, fundamental groups of complements of branch curves).

The courses by Catanese and Manetti are devoted instead to a similar question, a conjecture raised by Friedman and Morgan in the 1980s on the grounds of gauge theoretic speculations.

This conjecture is summarized by the acronym def=diff, and stated that diffeomorphic algebraic surfaces should be deformation equivalent. The course by Catanese reports briefly on the first nontrivial counterexamples, obtained by Manetti in the interesting case of surfaces of general type, and on the 'trivial' counterexamples, obtained independently by several authors, where a surface S is not deformation equivalent to the complex conjugate surface.

The focus in both courses is set on the simplest and strongest counterexamples, the so-called 'abc' surfaces, which are simply connected, and for which Catanese and Wajnryb showed that the diffeomorphism type is determined by the integers (a+c) and b. The course by Manetti focuses on the deformation theoretic and degeneration aspects (especially, smoothings of singularities), which up to now were scattered in a long series of articles by Catanese and Manetti. Catanese's course has a broader content and includes also other introductory facts on surfaces of general type and on singularities.

We would like to point out that, in spite of tremendous progress in 4-manifold topology, starting with Michael Freedman's solution of the problem of understanding the topology of simply connected 4-manifolds up to homeomorphism, and Simon Donaldson's gauge theoretic discoveries that smooth and topological structures differ drastically in dimension 4, the differential and symplectic topology even of simply connected symplectic 4-manifolds and algebraic surfaces is still a deep mystery.

Modern approaches to the study of symplectic 4-manifolds and algebraic surfaces combine a wide range of techniques and sources of inspiration. Gauge theory, symplectic geometry, pseudoholomorphic curves, singularity theory, moduli spaces, braid groups, monodromy, in addition to classical topology and algebraic geometry, combine to make this one of the most vibrant and active areas of research in mathematics.

Some keywords for the present volume are therefore pseudoholomorphic curves, algebraic and symplectic Lefschetz pencils, Dehn twists and monodromy, symplectic invariants, deformation theory and singularities, classification and moduli spaces of algebraic surfaces of general type, applications to mirror symmetry.

It is our hope that these texts will be useful to people working in related areas of mathematics and will become standard references on these topics.

We take this opportunity to thank the C.I.M.E. foundation for making the event possible, the authors for their hard work, the other lecturers for their interesting contributions, and the participants of the conference for their lively interest and enthusiastic collaboration.

We would also like to take this opportunity to thank once more the other authors for their work and apologize for the delay in publication of the volume.

May, 2007

Fabrizio Catanese
Gang Tian

Contents